MAT267: 3rd HW assignment. Due by 11:59pm on March 19.

1. (10 pt) Remember the Picard-Lindelöf Theorem? The existence and uniqueness theorem for $y' = f(x, y)$ where we assumed f is continuous in x and Lipschitz in y? (The one whose proof involved Picard iterates/the Picard map.) The proof I presented was for a single ODE. Now that we're considering systems of ODEs, we need an existence and uniqueness theorem for $\vec{X} = \vec{F}(t, \vec{X})$. Write up a modification of the proof for one ODE so that it's a proof for a system of ODEs. If you get stuck, see pages 387-394 in Hirsch, Smale, & Devaney — they do it for $\vec{X} = \vec{F}(\vec{X})$. Figure out how to modify their proof for autonomous systems (no t-dependence in the vector field) to make a proof for nonautonomous systems (the vector field can depend on t).

2. (10 pt) Consider $\vec{X}' = \vec{F}(t, \vec{X})$ with $\vec{X}(0) = \vec{X}_0$. $\vec{F} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$.

   Assume $\vec{F}$ is continuous in t and $\vec{X}$ and that $\vec{F}$ is C^1 in $\vec{X}$. (From class, this implies that given a compact set $K \subset \mathbb{R}^n$ there's a Lipschitz constant $L_K < \infty$ so that $\|\vec{F}(t, \vec{X}) - \vec{F}(t, \vec{Y})\| \leq L_K \|\vec{X} - \vec{Y}\|$ for all $\vec{X}, \vec{Y} \in K$.) Further assume that $\vec{F}(t, \vec{X})$ is bounded on compact sets in $\vec{X}$: given a compact set $K \subset \mathbb{R}^n$, there exists M_K so that $\|\vec{F}(t, \vec{X})\| \leq M_K$ for all t and all $\vec{X} \in K$.

   Assume that $\vec{X}(t)$ solves the initial value problem on some time interval (a, b). Assume that there's a compact set K so that $\vec{X}(t) \in K$ for all $t \in (a, b)$. Show that there exists some $a_1 < a$ and $b_1 > b$ so that $\vec{X}(t)$ solves the initial value problem on (a_1, b_1).

   You've extended the solution to a slightly larger interval of time. Give a condition that would allow you to apply the above procedure over and over again, extending the solution to (a_1, ∞). Give a condition that would allow you to apply the procedure over and over again, extending the solution to $(-\infty, b_1)$. Give a condition that would allow you to extend the solution to $(-\infty, \infty)$.

3. (5 pt) Consider $\vec{X}' = \vec{F}(\vec{X})$ where $\vec{F}$ is C^1. Assume there's a C^1 function $E : \mathbb{R}^n \to \mathbb{R}$ so that

   $$\vec{X}'(t) = \vec{F}(\vec{X}(t)) \qquad \Longrightarrow \qquad \frac{d}{dt} E(\vec{X}(t)) = 0.$$

   It follows that a solution of the initial value problem with $\vec{X}(0) = \vec{X}_0$ will satisfy $E(\vec{X}(t)) = E(\vec{X}_0)$ for all $t \in (\alpha, \beta)$ where (α, β) the interval from the existence and uniqueness theorem. Give a condition on the level sets of E that will guarantee that the solution $\vec{X}(t)$ exists for all time.

4. (5 pt) Consider $\vec{X}' = \vec{F}(\vec{X})$ where $\vec{F}$ is C^1. Assume there's a C^1 function $E : \mathbb{R}^n \to \mathbb{R}$ so that

$$\vec{X}'(t) = \vec{F}(\vec{X}(t)) \qquad \implies \qquad \frac{d}{dt} E(\vec{X}(t)) \leq 0.$$

It follows that a solution of the initial value problem with $\vec{X}(0) = \vec{X}_0$ will satisfy $E(\vec{X}(t)) \leq E(\vec{X}_0)$ for all $t \in [0, \beta)$ where (α, β) the interval from the existence and uniqueness theorem. Give a condition on the level sets of E that will guarantee that the solution $\vec{X}(t)$ exists on (α, ∞).

Assume the smooth dynamical system $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a flow for the for the system of ODEs

$$\vec{X}' = \vec{F}(\vec{X}), \qquad \vec{X}(0) = \vec{X}_0.$$

Recall that $\phi_t(\vec{X}_0) := \phi(t, \vec{X}_0)$ and

$$\frac{d}{dt} \phi_t(\vec{X}) = \vec{F}(\phi_t(\vec{X}))$$

for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$.

5. (10 pt) Let $v : \mathbb{R}^n \to \mathbb{R}$ be C^1 and assume v vanishes outside some compact set $K \subset \mathbb{R}^n$. We can "flow" this function v by defining

$$v_t(\vec{X}) = v(\phi_t(\vec{X})).$$

Assume v is identically one on the square of width .1, centered at the point $(1, 0)$ and v is identically zero outside the square of width .2 centered at the point $(1, 0)$. Consider the vector field $\vec{F}(\vec{X}) = A\vec{X}$ where A is a 2×2 matrix.

   (a) Let $A = (0, 1; -1, 0)$. What is $\phi_t(\vec{X})$? The function $v_t(\vec{X})$ will be identically one on the image (under the flow) of that square of width .1 and will be identically zero outside the image (under the flow) of that square of width .2. Sketch the image of the square of width .1 after it has flowed for $t = 1$. What is its area? Now consider the same v but shifted so that the square of width .1 has corners $(0, 0)$, $(.1, 0)$, $(.1, .1)$, and $(0, .1)$.

   (b) Repeat the above with $A = (2, 0; 0, -1)$ and with $A = (2, 1; 0, 2)$.

6. (10 pt) Show that

$$\frac{d}{dt} v_t(\vec{X}) = \vec{F}(\phi_t(\vec{X})) \cdot \nabla v(\phi_t(\vec{X})).$$

Show that this implies

$$\frac{d}{ds} v(\phi_s(\vec{X})) \bigg|_{s=0} = \vec{F}(\vec{X}) \cdot \nabla v(\vec{X}).$$

Show that

$$\frac{d}{dt} v_t(\vec{X}) = \frac{d}{ds} v_t(\phi_s(\vec{X})) \bigg|_{s=0}$$

and conclude that

$$\frac{d}{dt} v_t(\vec{X}) = \vec{F}(\vec{X}) \cdot \nabla v_t(\vec{X}).$$

I don't think you've learnt the divergence theorem yet in MAT257 but when you do you'll see that it implies

$$\frac{d}{dt}\int_{\mathbb{R}^n} v(\phi_t(\vec{X}))\, d\vec{X} = \int_{\mathbb{R}^n} \vec{F}(\vec{X}) \cdot \nabla v_t(\vec{X})\, d\vec{X} = -\int_{\mathbb{R}^n} div\vec{F}(\vec{X})\, v(\phi_t(\vec{X}))\, d\vec{X} \qquad (1)$$

where $div\vec{F}$ is the divergence of the vector field

$$div\vec{F}(\vec{X}) := \frac{\partial F_1}{\partial x_1}(\vec{X}) + \cdots + \frac{\partial F_n}{\partial x_n}(\vec{X})$$

Given a set $B \subset \mathbb{R}^n$ we define its characteristic function

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

If B is nonempty then χ_B is a discontinuous function. However, if B is "nice enough" (for example, smoothly bounded and closed) there exists a sequence of smooth functions $0 \le v_\epsilon \le 1$ so that $v_\epsilon(x) = 1$ for $x \in B$ and $v_\epsilon(x) = 0$ for $x \in \mathbb{R}^n - B_\epsilon$ were $B_\epsilon := \{x \,|\, dist(x, B) \le \epsilon\}$. Applying (1) to v_ϵ and taking $\epsilon \to 0$ in (1) yields

$$\frac{d}{dt}\text{Volume of } \phi_t(B) = \int_{\phi_t(B)} div\vec{F}(\vec{X})\, d\vec{X}. \qquad (2)$$

7. (10 pt) Consider $\vec{X}' = A\vec{X}$ and take B to be the parallelogram with one corner at $\vec{0}$ and with adjoining sides $\vec{X}_1$ and $\vec{X}_2$ where $\vec{X}_1$ and $\vec{X}_2$ are linearly independent.
   (a) What is the area of B at time $t = 0$? What is $\phi_t(B)$? What is the area of $\phi_t(B)$?
   (b) From exercise 7 of the previous homework assignment, what ODE should the area of $\phi_t(B)$ satisfy?
   (c) Assuming that (2) can be generalised to B that have piecewise smooth boundaries, what is (2) for this B and $\phi_t(B)$?

8. (10 pt) Consider the pendulum

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \vec{F}(\vec{X}) = \begin{pmatrix} y \\ -\frac{g}{L}\sin(x) - \frac{\alpha}{m}y \end{pmatrix}$$

   (a) What is $div\vec{F}(\vec{X})$ if there is no friction ($\alpha = 0$)? What does this mean about how the area of a "blob" changes as the blob is carried around by the flow?
   (b) What is $div\vec{F}(\vec{X})$ if there is friction ($\alpha > 0$)? What does this mean about how the area of a "blob" changes as the blob is carried around by the flow?