MAT267: 2nd HW assignment. Assigned March 5, due by 5pm on March 12. Please submit your HW to crowdmark.

1. (10 points) You know how to solve an initial value problem $\vec{X}' = A\vec{X}$ with $\vec{X}(0) = \vec{X}_0$. You compute e^{tA} and then $\vec{X}(t) = e^{tA}\vec{X}_0$ solves the initial value problem. For simplicity, we'll consider 2×2 matrices but everything below applies for $n \times n$ matrices.

   (a) Choose a basis $\{\vec{X}_1(0), \vec{X}_2(0)\}$ for $\mathbb{R}^2$ so that every initial data $\vec{X}_0$ can be written as a linear combination of $\vec{X}_1(0)$ and $\vec{X}_2(0)$. Construct two solutions $\vec{X}_1(t)$ and $\vec{X}_2(t)$ and demonstrate that every IVP can be written as a linear combination of these two solutions.

   (b) Define the solution matrix
   $$\mathcal{X}(t) = \left(\vec{X}_1(t) | \vec{X}_2(t)\right).$$
   Given an initial value problem, express the solution $\vec{X}(t)$ in terms of $\vec{X}_0$ and the solution matrix $\mathcal{X}(t)$. *It's not that your expression may only involve $\mathcal{X}(t)$; it can also involve evaluating that matrix at whatever time you want, it can involve powers of that matrix (evaluated at any time you want), inverses of that matrix (evaluated at any time you want), the determinant of that matrix (evaluated at any time you want), etc etc.*

   (c) Relate $\mathcal{X}(t)$ to e^{tA}. Prove that $\mathcal{X}(t)$ is invertible at all times.

   (d) Show that you can use $\mathcal{X}(t)$ to solve an initial value problem posed at some time other than time zero: $\vec{X}(t_0)$. Write the solution $\vec{X}(t)$ in terms of $\vec{X}(t_0)$ and the solution matrix $\mathcal{X}(t)$. (*See the previous italicized note.*)

2. (10 points) Assume $\vec{X}' = A(t)\vec{X}$ where $A(t)$ is a 2×2 time-dependent matrix. Let $\mathcal{X}(t)$ be any 2×2 matrix such that the columns of $\mathcal{X}(t)$ are solutions of $\vec{X}' = A(t)\vec{X}$. Assume each column is a solution on the time interval $(a, b) \subset \mathbb{R}$. Find an ODE that's satisfied by the determinant of $\mathcal{X}(t)$. Solve the ODE. Conclude that if $\mathcal{X}(t_0)$ is invertible for some t_0 then $\mathcal{X}(t)$ is invertible for all $t \in (a, b)$. *You can't use any e^{tA} stuff in your argument — that only holds for constant matrices. This is a problem that you should do directly using that $\vec{X}_1(t)$ and $\vec{X}_2(t)$ are solutions of $\vec{X}' = A(t)\vec{X}$.*

3. (10 points) Consider the ODE $x'' + a_1(t)\,x' + a_0(t)\,x = 0$. Assume that you've found two solutions $x_1(t)$ and $x_2(t)$ and that you've verified that you can solve every initial value problem $x(0) = x_0$, $x'(0) = v_0$ using a linear combination of them. Define the matrix
   $$\mathcal{X}(t) = \begin{pmatrix} x_1(t) & x_2(t) \\ v_1(t) & v_2(t) \end{pmatrix}$$
   where $v_1(t) = x_1'(t)$ and $v_2(t) = x_2'(t)$. Let $0 \in (a, b) \subset \mathbb{R}$ be its interval of definition.

   (a) Show that $\mathcal{X}(0)$ is invertible. *If you learnt and forgot Kramer's rule from linear algebra, now might be a good time to resurrect it and see that you can write c_1 and c_2 explicitly.*

(b) Find an ODE that's satisfied by the determinant of $\mathcal{X}(t)$. (Do this using only $x'' + a_1(t)\,x' + a_0(t)\,x = 0$; do not do it by writing this as an $\vec{X}' = A(t)\vec{X}$ system.) Solve the ODE. Conclude that $\mathcal{X}(t)$ is invertible on (a, b). Remind yourself why this means that you therefore use $\mathcal{X}(t)$ to solve an initial value problem posed at any time t_0.

4. (5 points) Consider the boundary value problem

$$x'' - 25x = 0, \qquad x(0) = 0, \qquad x(L) = 0.$$

Clearly $x(t) = 0$ is a solution. Can there be any other solutions? If no, prove why not. If yes, under what conditions? What if the boundary conditions were $x'(0) = x'(L) = 0$? If there are nonzero solutions (with either set of boundary conditions) why doesn't this violate our existence and uniqueness theorem?

5. (5 points) Consider the boundary value problem

$$x'' + 25x = 0, \qquad x(0) = 0, \qquad x(L) = 0.$$

Clearly $x(t) = 0$ is a solution. Can there be any other solutions? If no, prove why not. If yes, under what conditions? What if the boundary conditions were $x'(0) = x'(L) = 0$? If there are nonzero solutions (with either set of boundary conditions) why doesn't this violate our existence and uniqueness theorem?

6. (10 points) Find the solution of

$$x'' + 2x' + 2x = 3\sin(\omega t), \qquad x(0) = x0, \quad x'(0) = v_0.$$

(a) If someone were to say, "after some period of time, the initial data has been effectively forgotten" what are they talking about? What's left of the solution after this "period of time"?

(b) How would you choose ω so that this leftover (the "response of the system to the forcing") has largest possible magnitude?

(c) Set $\omega = 3$, $x_0 = 1$, and $v_0 = 0$ in your solution and plot x versus t. (Modify `https://www.desmos.com/calculator/yscrsftoan` and click on the box with an arrow in it to the right of "sign in" to save an image.) Also, compute $v(t)$ and do a phase plane plot. (Modify `https://www.desmos.com/calculator/ayqldhe6yp`.)

7. (15 points)

(a) Let A be an $n \times n$ matrix with real entries. Show that as $s \to 0$

$$det(I + sA) = (1 + s\,A_{1,1}) \cdot \cdots \cdot (1 + s\,A_{n,n}) + \mathcal{O}(s^2) = 1 + s\,Tr(A) + \mathcal{O}(s^2)$$

hence

$$\frac{d}{ds} det(I + sA)\bigg|_{s=0} = Tr(A).$$

Hint: Prove this by induction. For the 2×2 case, use

$$\begin{aligned} det(A+B) &= det(A_1 + B_1 | A_2 + B_2) = det(A_1|A_2 + B_2) + det(B_1|A_2 + B_2) \\ &= det(A_1|A_2) + det(A_1|B_2) + det(B_1|A_2) + det(B_1|B_2) \end{aligned}$$

where A_i denotes the ith column of A and B_i denotes the ith column of B.

(b) Let $B(s)$ be a smooth, matrix-valued function of s with $B(0) = I$. Use the previous exercise to show that

$$\frac{d}{ds} det(B(s))\bigg|_{s=0} = Tr(B'(0)).$$

(c) Let $C(t)$ be a smooth matrix-valued function and assume $C(0)$ is invertible. Use $B(s) = C(0)^{-1}\,C(s)$ to conclude

$$\frac{d}{ds} det(C(s))\bigg|_{s=0} = det(C(0))\,Tr(\,C(0)^{-1}\,C'(0)\,).$$

Generalize this away from derivatives evaluated at 0; do this by writing $C(s) = M(t+s)$ and use the above. If $M(t)$ is a smooth matrix-valued function and $M(t)$ is invertible, what is the right-hand-side of the ODE

$$\frac{d}{dt} det(M(t)) = ???$$

(d) Take $\mathcal{X}(t)$ from exercise 2 as the smooth matrix-valued function $C(t)$ above. Define the Wronskian $W(t) = det(\mathcal{X}(t))$ Prove that

$$W'(t) = Tr(A(t))\,W(t).$$

You may need to recall that if two matrices are similar then not only do they have the same determinant and eigenvalues but they also have the same trace.