2.2 Osgood's uniqueness theorem.

At the end of Section 1.2, we saw a situation in which it was possible to have infinitely many solutions to a differential equation $y' = f(x,y)$ through a point (x_0, y_0). In that example,

$$\begin{cases} y' = \sqrt{|y|} \\ y(x_0) = 0 \end{cases}$$

there was the constant solution ($y(x) = 0$) but there were also negative solutions that could "reach zero in finite time" and positive solutions that could "come from zero in the finite past". These solutions could be patched onto the zero solution, thus constructing infinitely many solutions to the initial value problem.

In the next result, we consider $y' = f(x,y)$ with initial condition $y(x_0) = y_0$. We give a condition on $f(x,y)$ which implies that there can be no more than one solution to the initial value problem. (It doesn't guarantee that such a solution exists, however.)

Theorem 3 (Osgood's Uniqueness Theorem)**.** *Consider the initial value problem*

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} .$$

Let $\mathscr{D} \subset \mathbb{R}^2$ be an open set containing (x_0, y_0). Assume that for all $(x,y_1),(x,y_2) \in \mathscr{D}$,

$$|f(x,y_1) - f(x,y_2)| \leq \varphi(|y_1 - y_2|), \tag{2.2.1}$$

for some continuous function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(u) > 0$ for $u > 0$ and $\varphi(0) = 0$ and

$$\int_0^1 \frac{du}{\varphi(u)} = +\infty. \tag{2.2.2}$$

Then no more than one solution passes through (x_0, y_0).

Notice that here we do not assume that f is continuous <u>in x</u>, and so we cannot invoke Peano's theorem to know that a solution exists.

Example. (*Lipschitz condition*) The most common choice of φ is $\varphi(u) = Ku$ for $K > 0$, i.e.

$$|f(x,y_1) - f(x,y_2)| \leq K|y_1 - y_2|.$$

This means that f is Lipschitz in the second variable y for any fixed x. Notice that

$$\int_0^1 \frac{du}{u} = \lim_{\varepsilon \downarrow 0} \ln u \Big|_{\varepsilon}^{1} = +\infty,$$

and so if $f(x,y)$ is Lipschitz in y then Osgood's Uniqueness theorem ensures the uniqueness of solutions, should they exist. If $|\frac{\partial f}{\partial y}|$ is bounded on the domain $\mathscr{D}$ by a constant K then the Lipschitz condition in y is guaranteed by the Mean Value Theorem. For example,

$$\frac{dy}{dx} = x^2 + y^2$$

will have a unique solution passing through any point (x_0, y_0). The existence follows from Peano's theorem.

Note that $\varphi(u) = \sqrt{u}$ satisfies the requirements of continuity, $u > 0 \implies \varphi(u) > 0$, and $\varphi(0) = 0$ but doesn't satisfy (2.2.2). Indeed, with this choice of φ, the ODE $y' = f(x,y) = \sqrt{|y|}$ satisfies the first requirement (2.2.1) of Osgood's Uniqueness theorem — and so we see that the key property for uniqueness will be (2.2.2).

Satisfying (2.2.1) means that ϕ is a "modulus of continuity" (in y) for f. In general, $f : \mathscr{D} \subset \mathbb{R}^n \to \mathbb{R}^k$ is uniformly continuous on $\mathscr{D}$ if and only if f has a modulus of continuity on $\mathscr{D}$. When we think of the $\varepsilon - \delta$ definition of continuity at a point x, we have that δ depends on ε and on x. What a modulus of continuity does is it allows you to run this in reverse: given a $\delta > 0$, it gives you ε (which depends on δ) so that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon(\delta)$.

Proof of Osgood's Uniqueness Theorem. We shall proceed by contradiction. Suppose there are two distinct solutions, $y_1(x)$ and $y_2(x)$ on $(\alpha, \beta) \ni x_0$ with $y_1(x_0) = y_2(x_0) = y_0$. We introduce their difference $z(x) := y_1(x) - y_2(x)$ and consider the initial value problem z solves:

$$\begin{cases} \frac{dz}{dx} = f(x, y_1(x)) - f(x, y_2(x)) \\ z(x_0) = 0 \end{cases} .$$

By the assumption (2.2.1),

$$|f(x, y_1(x)) - f(x, y_2(x))| \leq \varphi(|y_1(x) - y_2(x)|)$$

and, therefore,

$$z(x) \neq 0 \Longrightarrow \frac{dz}{dx} = f(x, y_1(x)) - f(x, y_2(x)) \leq \varphi(|z(x)|) < 2\varphi(|z(x)|). \tag{2.2.3}$$

Our strategy will be to use a "comparison argument" in which we introduce a well-chosen ODE and compare the solution of that ODE to $z(t)$.

Because the solutions $y_1(x)$ and $y_2(x)$ are not identical, they must differ at some point $x_1 \in (\alpha, \beta)$. We'll first assume that $x_1 > x_0$ and that $y_1(x_1) > y_2(x_1)$ and hence $z(x_1) > 0$. Let $v(x)$ be the solution of the initial value problem

$$\begin{cases} \frac{dv}{dx} = 2\varphi(v) \\ v(x_1) = z(x_1) =: z_1 > 0 \end{cases} . \tag{2.2.4}$$

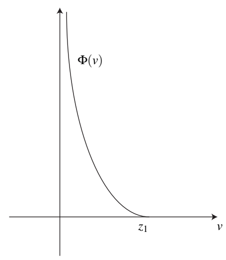
The ODE is separable and so we can solve it, determining $v(x)$ implicitly:

$$\Phi(v(x)) = \int_{v(x)}^{z_1} \frac{dv}{\varphi(v)} = \int_x^{x_1} 2\,dx = 2(x_1 - x) \quad \text{where} \quad \Phi(v) := \int_v^{z_1} \frac{du}{\varphi(u)}. \tag{2.2.5}$$

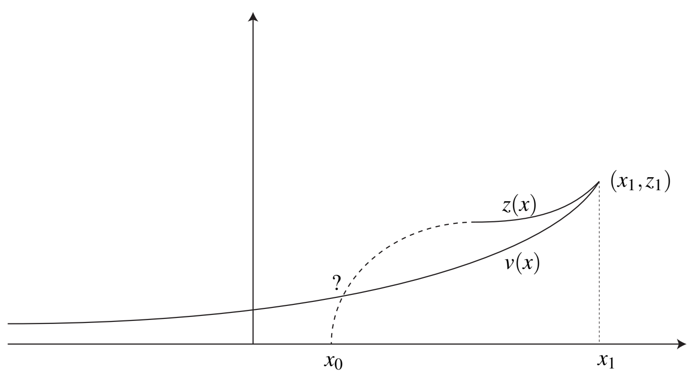
(*Pause and think carefully about this! Is this function $x \to v(x)$ well defined? Given x and x_1, the right-hand side is fixed. But could there be two different numbers v_1 and v_2 so that $\int_{v_i}^{z_1} 1/\varphi(v)\,dv = 2(x_1 - x)$?*) This solution is defined in an interval containing x_1; we now argue that it can be extended to be defined for all $x < x_1$.

We know φ is continuous and positive on $(0, \infty)$ hence $\Phi(z_1) = 0$. By the assumption (2.2.2), $\Phi(v) \uparrow +\infty$ as $v \downarrow 0$ and so $\Phi : (0, z_1] \to [0, \infty)$. Because $\Phi'(v) = -1/\varphi(v) < 0$, Φ is an invertible function with

$\Phi^{-1} : [0,\infty) \to (0,z_1]$. The graph of Φ is below. (Note: the graph is drawn to be concave up; this would happen if φ is a nondecreasing function. But we didn't assume this about φ and so, in principle, the graph of Φ could have different convexity.)



By definition, $v(x) = \Phi^{-1}(2(x_1 - x))$. The domain of Φ^{-1} is $[0,\infty)$ and so $v(x)$ is defined for all $x \le x_1$. Furthermore, v is an increasing, positive function on $(-\infty,x_1]$ and $v(x)$ decreases to 0 as x decreases to $-\infty$. See the graph of v in the figure below. (Again, we don't know that v is concave up unless we know that φ is nondecreasing. So the graph of v could have different concavity.) The fact that $v(x)$ is strictly positive on $(-\infty,x_1)$ (cannot "come from zero in the finite past") will be key in the argument that follows.



We now compare the two solutions $z(x)$ and $v(x)$. By (2.2.3), we know that at any point where the graphs of v and z coincide we must have

$$z(\tilde{x}) = v(\tilde{x}) > 0 \quad \Longrightarrow \quad z'(\tilde{x}) < v'(\tilde{x}).$$

This means that at any point $(\tilde{x}, z(\tilde{x})$ where the graphs meet the graph of z must be above the graph of v immediately to the left of $\tilde{x}$ and the graph of z must be below the graph of v immediately to the right of $\tilde{x}$. Specifically, there's some ε so that the graphs behave as claimed on $(\tilde{x}-\varepsilon, \tilde{x}+\varepsilon)$. (*Pause and think about this. Would this be true if all we knew was "$z(\tilde{x}) = v(\tilde{x}) > 0 \Longrightarrow z'(\tilde{x}) \le v'(\tilde{x}$"?)*) It follows that the graphs of v and z cannot meet twice in $(\alpha, x_1]$.

By construction, $z(x_1) = v(x_1)$ and so x_1 is a point at which the graphs meet and so the graph of z is above the graph of v immediately to the left of x_1. On the other hand, $z(x_0) = 0$ while v is strictly positive on $(-\infty,x_1]$. This means the graphs of z and v must cross at some point x_2 between x_0 and x_1. (Specifically, $x_2 := \inf\{x < x_1 | z(x) > v(x)\}$; the continuity of z and v implies that $z(x_2) = v(x_2)$. If such an x_2 existed, it'd be marked by the question mark in the figure.) But such an x_2 cannot exist because the graphs of z and v cannot meet twice in $(\alpha, x_1]$.

This contradiction implies that there cannot be a point $x_1 \in (\alpha,\beta)$ with $x_1 > x_0$ where $y_1(x_1) > y_2(x_1)$. Minor modifications of the above argument show that there cannot be a point $x_1 \in (\alpha,\beta)$ with $x_1 > x_0$ where $y_1(x_1) < y_2(x_1)$. And that there cannnot be a point in $x_1 \in (\alpha,\beta)$ with $x_1 < x_0$ where $y_1(x_1) \neq y_2(x_1)$. (*Make sure you understand how the argument would be modified for these three cases!*) Therefore $y_1(x) = y_2(x)$ for all $x \in (\alpha,\beta)$, as desired.

□

This proof relies on a comparison argument in which not-well-understood solution can be controlled in some way using a well-understood solution of a somehow-related problem. It's a classical argument that shows up in ODEs and in some PDEs.

We studied $v' = 2\varphi(v)$. In fact, the key things we needed were $2 > 1$ (so that we get the strict inequality $z'(\tilde{x}) < v'(\tilde{x})$ whenever $z(\tilde{x}) = v(\tilde{x})$) and that $2 > 0$ (so that $v(x) \downarrow 0$ as $x \downarrow -\infty$).

It was key that $v(x) > 0$ on $(-\infty, x_1]$. If it were possible for v to reach zero then this might happen at some point in (x_0, x_1) in which case the graph of z could happily stay above the graph of v while still satisfying $z(x_0) = 0$.

If the function φ were also assumed to be nondecreasing then one can prove the theorem in a completely different manner. See, for example, pages 12-13 of "Uniqueness and Nonuniqueness Criteria for Ordinary Differential Equations" by R.P. Agarwal and V. Lakshmikanthan. (Turn the page!) Note that the function φ from our statement of the theorem could be used to create a nondecreasing function $\overline{\varphi}$ via

$$\overline{\varphi}(u) := \sup_{\tilde{u}\in[0,u]} \varphi(\tilde{u})$$

but would the necessary requirement $\int_0^1 1/\overline{\varphi}(u)\,du = \infty$ still be true?

Exercise. We constructed v using that $v' = \varphi(v)$ is a separable equation. Can we extend v to all of $\mathbb{R}$? If we can't, what additional assumption on φ would allow us to?

Exercise. What is the interval (α,β)? If y_1 has maximal interval of existence (α_1,β_1) and y_2 has maximal interval of existence (α_2,β_2), does it follow that $(\alpha,\beta) = (\alpha_1,\beta_1) \cap (\alpha_2,\beta_2)$? Or could it be a proper subset?

Exercise. Suppose that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\varphi'(0)$ exists. Show that φ satisfies the condition of Osgood's theorem,

$$\int_0^c \frac{du}{\varphi(u)} = \infty.$$

from "Uniqueness and Nonuniqueness Criteria for ODEs" by Agarwal + Lakshmikantham

nondecreasing in y for each x in $0 \le x < \infty$. However, in Example 1.2.2 we have seen that (1.2.8) has a unique solution.

1.4 OSGOOD'S UNIQUENESS THEOREM

In this section we shall study a generalization of the Lipschitz uniqueness theorem which is due to Osgood . For this, we require the following :-

Lemma 1.4.1. Let $g(z)$ be a continuous and nondecreasing function in the interval $[0, \infty)$ and $g(0) = 0$, $g(z) > 0$ for $z > 0$. Also,

(1.4.1) $$\lim_{\epsilon \to 0^+} \int_\epsilon \frac{dz}{g(z)} = \infty.$$

Let $\phi(x)$ be a nonnegative continuous function in $[0, a]$. Then,

(1.4.2) $$\phi(x) \le \int_0^x g(\phi(t))dt, \ 0 < x \le a$$

implies that $\phi(x) = 0$ in $[0, a]$.

Proof. Define $\Phi(x) = \max_{0 \le t \le x} \phi(t)$ and assume that $\Phi(x) > 0$ for $0 < x \le a$. Then, $\phi(x) \le \Phi(x)$ and for each x there is an $x_1 \le x$ such that $\phi(x_1) = \Phi(x)$. From this, we have

$$\Phi(x) = \phi(x_1) \le \int_0^{x_1} g(\phi(t))dt \le \int_0^x g(\Phi(t))dt,$$

i.e., the increasing function $\Phi(x)$ satisfies the same inequality as $\phi(x)$ does. Let us set $\bar{\Phi}(x) = \int_0^x g(\Phi(t))dt$, then $\bar{\Phi}(0) = 0, \Phi(x) \le \bar{\Phi}(x)$ and $\bar{\Phi}'(x) = g(\Phi(x)) \le g(\bar{\Phi}(x))$. Hence, for $0 < \delta < a$, we have

$$\int_\delta^a \frac{\bar{\Phi}'(x)}{g(\bar{\Phi}(x))}dx \le a - \delta < a.$$

However, from (1.4.1), it follows that

$$\int_\delta^a \frac{\bar{\Phi}'(x)}{g(\bar{\Phi}(x))}dx = \int_\epsilon^\alpha \frac{dz}{g(z)}, \ \bar{\Phi}(\delta) = \epsilon, \bar{\Phi}(a) = \alpha$$

becomes infinite when $\epsilon \to 0^+ (\delta \to 0)$. This contradiction shows that $\Phi(x)$ cannot be positive and so $\Phi(x) \equiv 0$, and hence $\phi(x) = 0$ in $[0, a]$. ∎

Theorem 1.4.2 (Osgood's Uniqueness Theorem). Let $f(x, y)$ be continuous in $\bar{S}$ and for all $(x, y), (x, \bar{y}) \in \bar{S}$ it satisfies **Osgood's condition**

(1.4.3) $$|f(x, y) - f(x, \bar{y})| \le g(|y - \bar{y}|),$$

where $g(z)$ is the same as in Lemma 1.4.1. Then, the initial value problem (1.1.1) has at most one solution in $|x - x_0| \le a$.

Proof. Suppose $y(x)$ and $\bar{y}(x)$ are two solutions of (1.1.1) in $|x - x_0| \le a$. We shall show that $y(x) = \bar{y}(x)$ in $[x_0, x_0 + a]$. From (1.4.3) it follows that

$$\begin{aligned} |y(x_0 + x) - \bar{y}(x_0 + x)| &\le \int_{x_0}^{x_0+x} |f(t, y(t)) - f(t, \bar{y}(t))|dt \\ &\le \int_{x_0}^{x_0+x} g(|y(t) - \bar{y}(t)|)dt \\ &= \int_0^x g(|y(z + x_0) - \bar{y}(z + x_0)|)dz. \end{aligned}$$

For x in $[0, a]$, we set $\phi(x) = |y(x + x_0) - \bar{y}(x + x_0)|$. Then, the nonnegative continuous function $\phi(x)$ satisfies the inequality (1.4.2), and therefore, Lemma 1.4.1 implies that $\phi(x) = 0$ in $[0, a]$, i.e., $y(x) = \bar{y}(x)$ in $[x_0, x_0 + a]$. If x is in $[x_0 - a, x_0]$, then the proof remains the same except that we need to define the function $\phi(x) = |y(x_0 - x) - \bar{y}(x_0 - x)|$ in $[0, a]$. ∎

Example 1.4.1. Consider the initial value problem

(1.4.4) $$y' = Ly, \ y(0) = 0$$

where $L > 0$. For this problem, we choose $g(z) = Lz$, which is clearly continuous and nondecreasing in the interval $[0, \infty)$. Further, since $g(0) = 0$, $g(z) > 0$ for $z > 0$, and $\lim_{\epsilon \to 0^+} \int_\epsilon [g(z)]^{-1}dz = \frac{1}{L} \lim_{\epsilon \to 0^+} \ln \frac{1}{\epsilon} = \infty$. This function $g(z)$ satisfies the conditions of Lemma 1.4.1. Next, for any y and $\bar{y}$ we have

$$|f(x, y) - f(x, \bar{y})| = |Ly - L\bar{y}| = g(|y - \bar{y}|)$$

and hence Osgood's condition (1.4.3) is also satisfied. Therefore, from Theorem 1.4.2 the problem (1.4.4) has a unique solution, namely, $y(x) \equiv 0$. ∎