1.2 Equations with separable variables.

Suppose we would like to solve an equation of the form

$$\frac{dy}{dx} = f(x)g(y),$$

where f, g are continuous functions. If $g(y_0) = 0$ then $y(x) \equiv y_0$ is a solution (check!). So us let look for solutions where $g(y) \neq 0$. Formally, we can find solutions as follows:

$$\begin{aligned} \frac{dy}{g(y)} &= f(x)\,dx \\ \int \frac{dy}{g(y)} &= \int f(x)\,dx \\ G(y) &= F(x) + C, \end{aligned}$$

where G and F are antiderivatives, i.e.

$$G'(y) = \frac{1}{g(y)},\ F'(x) = f(x).$$

Finally, solve for y by taking inverse $y = G^{-1}(F(x)+C)$ on (a,b) if G^{-1} exists. How do we justify this calculation, especially, treating dx and dy are separate entities?

First of all, the first two steps of the above computation can be written as follows:

$$\frac{y'(x)}{g(y(x))} = f(x) \implies \int \frac{y'(x)\,dx}{g(y(x))} = \int f(x)\,dx$$

and making a change of variables we get $G(y(x)) = F(x) + C$ as before, so separating dx and dy corresponds to skipping this change of variables step, even though it might seem strange at first. Taking derivatives,

$$\frac{d}{dx} G(y(x)) = \frac{y'(x)}{g(y(x))} = f(x),$$

we see that the solution of the implicit equation we found, $G(y) = F(x) + C$, is a solution of the original differential equation.

The reason we can take inverse $y(x) = G^{-1}(F(x)+C)$ is because we are solving around some point (x_0, y_0) where $g(y_0) \neq 0$, so $g(y) \neq 0$ in a small neighbourhood. Therefore, $G'(y) = 1/g(y)$ will be either strictly positive or negative and $G(y)$ will be strictly increasing or decreasing. Of course, the solution passing through (x_0, y_0) corresponds to the choice of $C = G(y_0) - F(x_0)$.

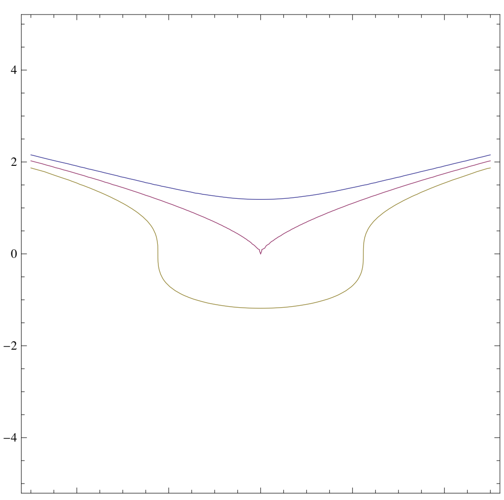
Remark. Notice that the above calculation shows by construction that, if f and g are continuous, the equation with separable variables has *unique* solution in a neighbourhood of any point (x_0, y_0) such that $g(y_0) \neq 0$. Later, when we discuss existence and uniqueness of solutions we should remember that sometimes existence and/or uniqueness follows by construction.

Remark. What's meant by "formally"? This means we're doing a sequence of manipulations of unclear validity; at the end of them we find a function $y(x)$. We can then check whether or not the function we found is a solution of the ODE $y' = f(x)g(y)$ and, if it is, what its interval of definition is. That said, the equation $-f(x)\,dx+dy/g(y)=0$ can be rigorously understood in terms of 1-forms on the manifold $\mathbb{R}^2$. A 1-form is a linear functional from the tangent space at a point on a Manifold to $\mathbb{R}$. Given a vector in the tangent space, the 1-form returns a number. The tangent space at a point in a manifold can be understood in various ways; the most useful way in this context is in terms of "germs". A "germ" is an equivalence class of parameterized curves through the point in the manifold. Specifically, if we're looking at the point $(x_0,y_0)\in\mathbb{R}^2$ then a tangent vector $\vec{v}\in T_{(x_0,y_0)}M$ is represented (by the equivalence class[1] of) all smooth curves $\gamma(t)$ in a neighbourhood of (x_0,y_0) such that $\gamma(0)=(x_0,y_0)$ and $\gamma'(0)=\vec{v}$. For $M=\mathbb{R}^2$, we have $\gamma(t)=<\gamma_1(t),\gamma_2(t)>$ and the 1-form $-f(x)\,dx+dy/g(y)$ acts on one of these curves (and hence on the tangent vector they represent) by producing the number $-f(\gamma_1(0))\,\gamma_1'(0)+\gamma_2'(0)/g(\gamma_2(0))$. The equation $-f(x)\,dx+dy/g(y)=0$ corresponds to seeking those curves such that $-f(\gamma_1(0))\,\gamma_1'(0)+\gamma_2'(0)/g(\gamma_2(0))=0$. To interpret $-f(x)\,dx+dy/g(y)=0$ in terms of ODEs, one seeks a curve $\gamma(t)$ such that for an interval of times (a,b) one has $-f(\gamma_1(t))\,\gamma_1'(t)+\gamma_2'(t)/g(\gamma_2(t))=0$ for all $t\in(a,b)$. If the curve $\gamma(t)=(t,y(t))$ then this corresponds to seeking a solution $y(x)$ on (a,b). If the curve $\gamma(t)=(x(t),t)$ then this corresponds to seeking a solution $x(y)$ on (a,b). For some online notes[2] that show germs, see pages 11-19 of Marco Billò's "notes on Differential Geometry". And, of course, you'll learn how to make the casual comments in this remark rigorous by taking a course on differential geometry.

Example. Consider equation $2x\,dx-9y^2\,dy=0$. As we mentioned above, one can think of this equation as one of the two equations:

$$\frac{dy}{dx}=\frac{2x}{9y^2}\ \text{ when }\ y\neq 0$$
$$\frac{dx}{dy}=\frac{9y^2}{2x}\ \text{ when }\ x\neq 0.$$

When $y=0$ solve the second equation, and when $x=0$ solve the first. In either case, $\int 2x\,dx=\int 9y^2\,dy$, and $x^2=3y^3+C$ is an implicit general solution. Below are the plots for $C=-5,0,5$.



[1] Two curves γ and $\tilde{\gamma}$ are equivalent if they satisfy $\gamma(0)=\tilde{\gamma}(0)$ and $\gamma'(0)=\tilde{\gamma}'(0)$.

[2] If you're reading these notes in a non-clickable way, check out `http://personalpages.to.infn.it/~billo/` and `http://www.math.toronto.edu/mpugh/Teaching/MAT267_19/differential_geometry_notes.pdf`.

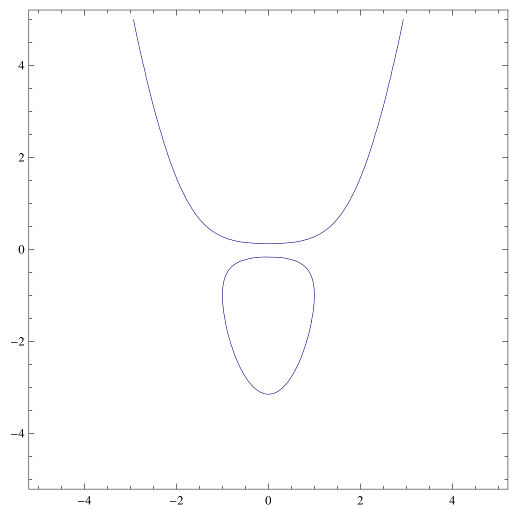
Example. Consider the equation

$$\frac{dy}{dx} = \frac{2xy}{1+y},\ y \neq -1.$$

This is an equation with separable variables and we can write

$$\left(\frac{1}{y}+1\right)dy = 2x\,dx \ \text{ if } \ y \neq 0.$$

As above, $y(x) \equiv 0$ is a solution of the original equation, so we should not discard it. For $y \neq 0$, integrating both sides yields the implicit solution $\ln|y| + y = x^2 + C$. Solving for y in this case is impossible, and you can see the plot for a typical C below:



When $y = -1$, this is exactly where the curve has vertical slope ($y' = \infty$), so we can think that at this point we are solving $\frac{dx}{dy} = \frac{1+y}{2xy}$ when $x \neq 0$. Also, remember that the line $y \equiv 0$ is also a solution.

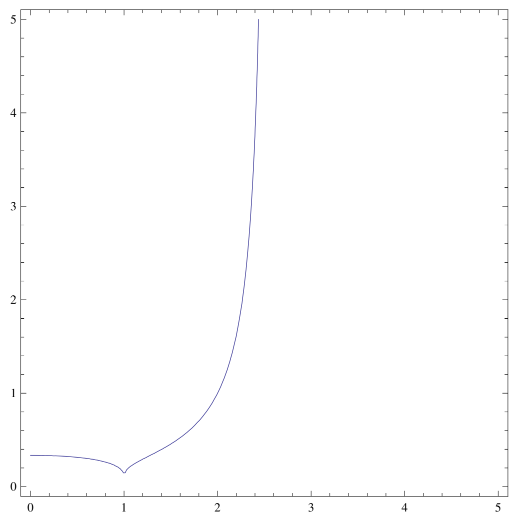
Example. Let us find a solution of $xy^2\,dx + (1-x)\,dy = 0$ satisfying $y(2) = 1$, i.e. passing through a point $(2,1)$. Let us separate the variables,

$$\left(\frac{x}{1-x}\right)dx + \frac{1}{y^2}\,dy = 0$$

assuming $x \neq 1, y \neq 0$. As before, notice that $y(x) \equiv 0$ is a solution if we rewrite this as equation for $y'(x)$ and $x(y) \equiv 1$ is a solution if we rewrite it as equation for $x'(y)$. For $x \neq 1, y \neq 0$ we integrate and we get

$$\int\left(\frac{1}{1-x}-1\right)dx + \int\frac{1}{y^2}\,dy = C \Longrightarrow \ln|1-x| + x + \frac{1}{y} = -C.$$

For $x = 2, y = 1$ we get $C = 3$ and thus $\ln|1-x| + x + \frac{1}{y} = 3$. Plot:



Important example. Let us consider equation

$$\frac{dy}{dx} = \sqrt{|y|}.$$

Again, $y(x) \equiv 0$ is a solution. Otherwise, we integrate:

$$\frac{dy}{\sqrt{y}} = dx \Longrightarrow 2\sqrt{y} = x - c \Longrightarrow y = \frac{1}{4}(x-c)^2, \quad x \in (c, \infty)$$

is a solution for any fixed $c \in \mathbb{R}$. Similarly, $y = -(x-c)^2/4$ with $x \in (-\infty, c)$ is a solution for any fixed $c \in \mathbb{R}$. These can be patched together to create infinitely many solutions

$$y_{ab}(x) = \begin{cases} -\frac{(x-a)^2}{4} & x < a \\ 0 & a \leq x \leq b \\ \frac{(x-b)^2}{4} & x > b \end{cases}, \qquad x \in \mathbb{R}$$

for any $a < b$. In this way, we see that the initial value problem

$$\begin{cases} y' = \sqrt{|y|} \\ y(0) = 0 \end{cases}$$

has infinitely many solutions (including the $y(x) = 0$ solution).

Remark. As we will see in the proof of Osgood's Uniqueness Theorem, the fundamental problem is as follows. If one considers initial data $y(x_0) = y_0 < 0$ then the solution "reaches zero in finite time": there's an $x_1 < \infty$ so that $y(x) \to 0$ and $y'(x) \to 0$ as $x \uparrow x_1$. This allows the solution to be continued with the zero solution. Similarly, if one considers $y(x_0) = y_0 > 0$ then the solution "comes from zero in the finite past": there's an $x_1 > -\infty$ so that $y(x) \to 0$ and $y'(x) \to 0$ as $x \downarrow x_1$ This allows the solution to have emerged from the zero solution.

Remark. In a similar manner, one can construct infinitely many solutions for

$$\begin{cases} y' = |y|^{\alpha} \\ y(0) = 0 \end{cases}$$

for $\alpha \in (0,1)$.