Complex eigenvalues, ellipses

Assume $\vec{X}' = A\vec{X}$ where A is 2×2, has real-valued entries and has eigenvalue - eigenvector pairs

$\lambda = \mu + i\nu$ $\qquad\qquad$ $\lambda = \mu - i\nu$

$\vec{v} = \vec{a} + i\vec{b}$ $\qquad\qquad$ $\vec{v} = \vec{a} - i\vec{b}$

Two complex-valued solutions

$$\vec{X}_1(t) = e^{(\mu+i\nu)t}(\vec{a} + i\vec{b})$$

$$\vec{X}_2(t) = e^{(\mu-i\nu)t}(\vec{a} - i\vec{b})$$

Any linear combination of solutions is a solution $\Rightarrow$ $\frac{1}{2}(\vec{X}_1 + \vec{X}_2)$ and $\frac{1}{2i}(\vec{X}_1 - \vec{X}_2)$ are solutions.

transl: $Re(\vec{X}_1(t))$ & $Im(\vec{X}_1(t))$ are real-valued solutions of $\vec{X}' = A\vec{X}$

$$\vec{X}_1(t) = e^{(\mu+i\nu)t}(\vec{a} + i\vec{b})$$

$$= e^{\mu t}(\cos(\nu t) + i\sin(\nu t))(\vec{a} + i\vec{b})$$

$$= \vec{u}(t) + i\vec{w}(t)$$

$$= e^{\mu t}[\cos(\nu t)\vec{a} - \sin(\nu t)\vec{b}]$$

$$+ i e^{\mu t}[\sin(\nu t)\vec{b} + \cos(\nu t)\vec{a}]$$

So $\vec{u}(t) = e^{\mu t}\underbrace{\left[\cos(\nu t)\vec{a} - \sin(\nu t)\vec{b}\,\right]}_{\text{Some oscillatory thing}}$

$\vec{w}(t) = e^{\mu t}\left[\cos(\nu t)\vec{b} + \sin(\nu t)\vec{a}\,\right]$

i.e. $\vec{u}(t)$ and $\vec{w}(t)$ are of the form "something constant ($\mu=0$)/growing ($\mu>0$)/decaying ($\mu<0$)" times "something oscillatory".

Our goal:

1) understand the "something oscillatory" and show it's an ellipse/circle that may (or may not) be rotated.
2) use this to understand solutions of the IVP with $\vec{x}(0) = \vec{x}_0$.

$\vec{u}(t) = e^{\mu t}\vec{E}_u(t) \quad \vec{w}(t) = e^{\mu t}\vec{E}_w(t)$

$\vec{E}_u(t) = \cos(\nu t)\vec{a} - \sin(\nu t)\vec{b}$

$\vec{E}_w(t) = \cos(\nu t)\vec{b} + \sin(\nu t)\vec{a}$

$\vec{E}_u(2\pi/\nu) = \vec{E}_u(0)$ and $\vec{E}_w(2\pi/\nu) = \vec{E}_w(0)$

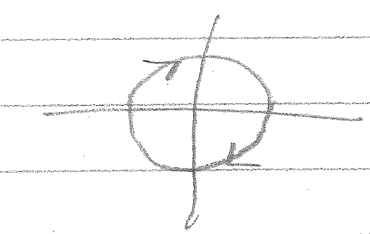
they're periodic w/ period $2\pi/\nu$.

$$\vec{E}_u\left(t-\frac{\pi}{2}\frac{1}{\nu}\right) = \cos\left(\nu t-\frac{\pi}{2}\right)\vec{a} - \sin\left(\nu t-\frac{\pi}{2}\right)\vec{b}$$
$$= \sin(\nu t)\,\vec{a} + \cos(\nu t)\,\vec{b}$$
$$= \vec{E}_w(t).$$

So $\vec{E}_u$ and $\vec{E}_w$ have the same trajectory; it's just that $\vec{E}_u(t)$ is $\frac{\pi}{2}\frac{1}{\nu}$ "ahead of" $\vec{E}_w(t)$ in time. $\Rightarrow$ It suffices to understand the trajectory of $\vec{E}_u(t)$.

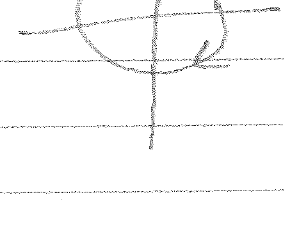
$$\vec{E}_u(t) = \cos(\nu t)\vec{a} - \sin(\nu t)\vec{b}$$
$$= \left(\vec{a}\,\middle|\,\vec{b}\right)\begin{pmatrix}\cos(\nu t)\\ -\sin(\nu t)\end{pmatrix}$$

We have the circle given by $\begin{pmatrix}\cos(\nu t)\\ -\sin(\nu t)\end{pmatrix}$

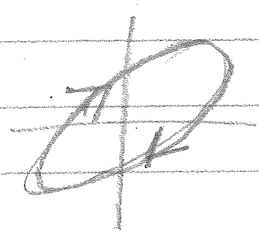


and it's being acted on by the linear transformation $T = \left(\vec{a}\,\middle|\,\vec{b}\right)$

before

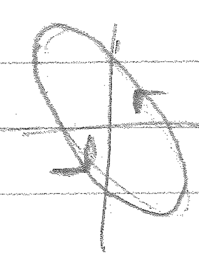


$\det(T)>0$



after

$\det(T)<0$



④

The sign of the determinant of T tells us if $\vec{E}_u(t)$ is clockwise or counterclockwise but what determines the shape of the ellipse and its tilt?

__Answer__ we need to compute the singular value decomposition of T.

$$T = (\vec{a}|\vec{b}) = U \Sigma V^T$$

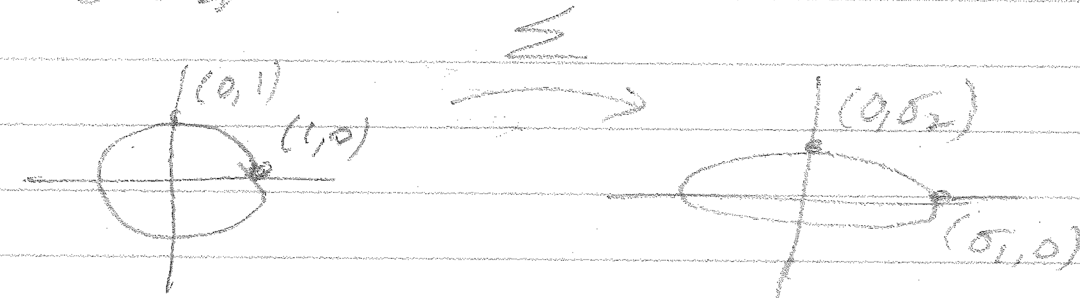
U and V are orthogonal matrices:

$$UU^T = U^T U = I$$

$$VV^T = V^T V = I$$

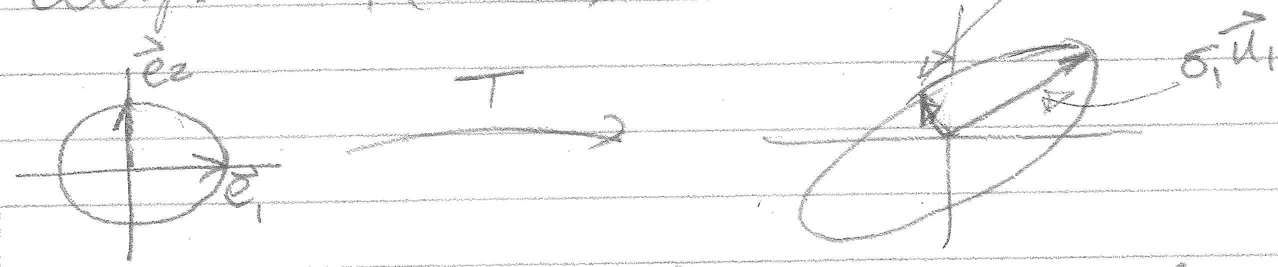
Σ is a diagonal matrix with positive entries — these are the "singular values" of T. (In general, the singular values are nonnegative — we have an invertible* matrix T and so they are positive.

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$



*Why? Think this through! ☺

What gives the orientation of the ellipse = T(circle)?



$U = (\vec{u}_1 | \vec{u}_2)$ are orthogonal unit vectors.

For a very nice geometric explanation of SVDs, see "We recommend a Singular Value Decomposition" by David Austin, posted on the AMS website.

The key idea: every 2x2 matrix has orthogonal vectors $\vec{v}_1, \vec{v}_2$ and $\vec{u}_1, \vec{u}_2$ so that

$$T\vec{v}_1 = \sigma_1 \vec{u}_1$$
$$T\vec{v}_2 = \sigma_2 \vec{u}_2$$

$$U = (\vec{u}_1 | \vec{u}_2) \quad \Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \quad V = (\vec{v}_1 | \vec{v}_2).$$

How to compute them?

1. Compute TT^T
2. Find eigenvalues + eigenvectors of TT^T
3. Normalize eigenvectors $\to U$.

build $\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}$

$$U = \left(\vec{u}_1 \,\middle|\, \vec{u}_2 \right)$$

normalized eigenvectors

~~to find V, find the eigenvectors of $T^T T$.~~

Note: The SVD is defined for any $n \times m$ matrix

$$\underbrace{A}_{n \times m} = \underbrace{U}_{n \times m} \underbrace{\Sigma}_{n \times m} \underbrace{V^T}_{m \times m}$$

Relevant for other compression (SVD)

AA^T is $n \times n$. Its eigenvalues are nonnegative and so they have square roots

A^TA is $m \times m$. Its eigenvalues are also nonnegative and so they have square roots too.

If $n = m$ then $\text{eig}(AA^T) = \text{eig}(A^TA)$

If $n > m$ then

$$\text{eig}(AA^T) = \text{eig}(A^TA) \cup \{n-m \text{ zero eigenvalues}\}$$

Compute $\sqrt{\text{eig}(AA^T)}$, sort in descending order and fill the diagonal of Σ. (Sorting is by convention.)

Okay! So we understand

$$\vec{u}(t) = e^{\mu t}\vec{E}_u(t)$$
$$= e^{\mu t}\left[\cos(\nu t)\vec{a} - \sin(\nu t)\vec{b}\right]$$

~~because $\vec{E}_u(t)$ is the ellipse that is the image of the unit circle under~~

$$T = (\vec{a}|\vec{b})$$

Now to understand the general initial value problem

$$\vec{X}(t) = c_1\vec{u}(t) + c_2\vec{w}(t)$$
$$\vec{X}(0) = c_1\vec{u}(0) + c_2\vec{w}(0)$$
$$= c_1\vec{a} + c_2\vec{b} = \vec{X}_0$$

$\vec{a}$ and $\vec{b}$ are Linearly independent and so $\exists !$ solution

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = T^{-1}\vec{X}_0$$

$$\vec{X}(t) = e^{\mu t}\left[c_1\vec{E}_u(t) + c_2\vec{E}_w(t)\right]$$

$$= e^{\mu t}\left[c_1\cos(\nu t)\vec{a} - c_1\sin(\nu t)\vec{b} + c_2\cos(\nu t)\vec{b} + c_2\sin(\nu t)\vec{a}\right]$$

$$= e^{\mu t}\left(c_1\vec{a} + c_2\vec{b} \,\middle|\, +c_1\vec{b} - c_2\vec{a}\right)\begin{pmatrix}\cos(\nu t) \\ -\sin(\nu t)\end{pmatrix}$$

So our solution of the IVP is of the form

$$e^{\mu t} \cdot \text{"something oscillatory"}$$

The oscillatory thing is

$$\left[c_1\vec{a}+c_2\vec{b} \,\middle|\, -c_2\vec{a}+c_1\vec{b} \right] \begin{pmatrix} \cos(\nu t) \\ -\sin(\nu t) \end{pmatrix}$$

$$= \left[\vec{a} \,\middle|\, \vec{b} \right] \begin{bmatrix} c_1 & -c_2 \\ c_2 & c_1 \end{bmatrix} \begin{pmatrix} \cos(\nu t) \\ -\sin(\nu t) \end{pmatrix}$$

We have $T_C \begin{pmatrix} \cos(\nu t) \\ -\sin(\nu t) \end{pmatrix}$

where $T_C := TC = \underbrace{\left[\vec{a} \,\middle|\, \vec{b} \right]}_{\text{from before}} \underbrace{\begin{bmatrix} c_1 & -c_2 \\ c_2 & c_1 \end{bmatrix}}_{\text{det'd by the initial conditions}}$

note $C^T C = C C^T$

$$= \begin{pmatrix} c_1^2+c_2^2 & 0 \\ 0 & c_1^2+c_2^2 \end{pmatrix} = |\vec{c}|^2 I$$

How is the ellipse from $\vec{E}_u(t)$ (determined by T) related to the ellipse for the IVP, determined by TC?

Let's compute the SVD of TC!

1. eigenvalues of
$$(TC)(TC)^T \;?$$

$$(TC)(TC)^T = (TC)(C^T T^T)$$
$$= T(CC^T)T^T$$
$$= T(|\vec{c}|^2 I)T^T$$
$$= |\vec{c}|^2 \, TT^T$$

$$\Rightarrow \sqrt{\text{eig}((TC)(TC)^T)} = |\vec{c}| \sqrt{\text{eig}(TT^T)}$$

$\Rightarrow$ singular values of TC equal $|\vec{c}|$ times the singular values of T.

transl: compute the magnitude of $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ where $c_1\vec{a} + c_2\vec{b} = \vec{X_0}$ and dilate or shrink the ellipse by $|\vec{c}|$. But wait...

2. Do we know that TC doesn't rotate the ellipse differently than T did? Let's check!

Need eigenvectors of

$(TC)(TC)^T$. How do they relate

to the eigenvectors of TT^T?

We know that

$$(TC)(TC)^T = |\vec{c}|^2 \, TT^T$$

and so they have the same eigenvectors!

$$T = U \Sigma V^T$$

same

$$TC = U(|\vec{c}|\Sigma)\tilde{V}^T$$

V and $\tilde{V}$ can be different because they come from the eigenvectors of

$$T^T T$$

and $C^T T^T C$ respectively.

Matlab program comparing $\vec{E}_u(t)$ to the ellipse from the SVD of $(\vec{a}|\vec{b})$.

```
clear

% define a matrix that has complex eigenvalues
A = [0 1; -4 -1];

% use this if I want to define a random matrix.  The following will often
% give a matrix that has complex eigenvalues...
% A = [0 1; -rand rand-1/2];

% compute the eigenvalues and eigenvectors of A
[v,d] = eig(A)
v1 = v(:,1);
% extract the real & imaginary parts of the first eigenvalue
mu = real(d(1,1));
nu = imag(d(2,2));
% extract the real and imaginary parts of the first eigenvector
a = real(v1);
b = imag(v1);
% build the matrix T
T = [a,b];

% I want to plot the circle and ellipse for t in [0, 2*pi/nu] so first I
% define a time array
N = 100;
dt = (2*pi/nu)/N;
t = 0:dt:2*pi/nu;
% I define the circle
C = [cos(nu*t);-sin(nu*t)];
% I define the ellipse which is T acting on the circle
E = T*C;

% let's plot the circle and ellipse!
figure(1)
clf
plot(C(1,:),C(2,:))
hold on
plot(E(1,:),E(2,:),'r')
axis('equal')

% now I want to find the major and minor axes of the ellipse.  So I compute
% the SVD of the matrix T.
[U,S,V] = svd(T);
% the first vector is the first column of U times the first singular value
U1 = U(:,1)*S(1,1);
% the second vector is the second column of U times the second singular value
U2 = U(:,2)*S(2,2);
% mark these points on the ellipse.  (I'm happy to see that they're on the
% ellipse and appear in the right location!)
plot(U1(1),U1(2),'bo')
plot(U2(1),U2(2),'bx')

% the following is just aesthetics to make it clearer that I really did
% find the closest and furtherest points from the origin...
x = -1:.01:1;
plot(U1(1)*x,U1(2)*x,':')
plot(U2(1)*x,U2(2)*x,':')
xx = 0:.01:1;
plot(U1(1)*xx,U1(2)*xx,'-')
plot(U2(1)*xx,U2(2)*xx,'-')
```