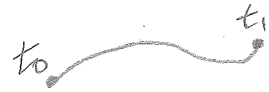
Proof of continuity in X_0, forward in time.

1. $\exists\ X(t)$ soln of $\begin{cases} \vec{X}' = \vec{F}(\vec{X}) \\ \vec{X}(t_0) = \vec{X}_0 \end{cases}$ on $[t_0, t_1]$.

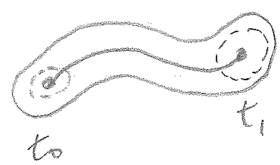


$\leftarrow \gamma = \{ \vec{x}(t) \mid t \in [t_0, t_1] \}$

compact set. Define $T_\varepsilon = \{ \vec{y} \in \mathbb{R}^n \mid \text{dist}(\vec{y}, \gamma) \le \varepsilon \}$

T_ε is closed + bounded $\Rightarrow$ compact.

$$M := \max_{y \in T_\varepsilon} |\vec{F}(\vec{y})| \qquad K = \text{Lipschitz const of } \vec{F}\big|_{T_\varepsilon}$$

2. 

Idea: show that $\vec{Y}(t) \in B_{\frac{\varepsilon}{2} e^{-K(t_1 - t)}}(\vec{X}(t))$ for all $t \in [t_0, t_1]$

define $\delta = \frac{\varepsilon}{2} e^{-K(t_1 - t_0)} < \frac{\varepsilon}{2} < \varepsilon$

Assume $\vec{Y}_0 \ni \|\vec{X}_0 - Y_0\| < \delta \Rightarrow \vec{Y}_0 \in B_{\frac{\varepsilon}{2}}(\vec{X}_0)$

$\Rightarrow B_{\frac{\varepsilon}{2}}(Y_0) \subset T_\varepsilon \Rightarrow |\vec{F}(\vec{z})| \le M \quad \forall z \in B_{\frac{\varepsilon}{2}}(\vec{Y}_0)$

and K works as a Lipschitz constant for $\vec{F}\big|_{B_{\frac{\varepsilon}{2}}(Y_0)}$.

3. By existence and uniqueness theorem, $\exists$ solution of $\begin{cases} Y' = F(Y) \\ Y(t_0) = Y_0 \end{cases}$ on $(t_0 - a, t_0 + a)$ $\quad a = \min\{ \frac{\varepsilon/2}{M}, \frac{1}{K} \}$

don't really need the $\frac{1}{K}$ if we use the other existence proof

$X(t)$ defined on $[t_0, t_1]$

$Y(t)$ defined on $(t_0 - a, t_0 + a)$

$$X(t) = X_0 + \int_{t_0}^{t} F(X(s))\,ds \quad \text{on } [t_0, t_1]$$

$$Y(t) = Y_0 + \int_{t_0}^{t} F(Y(s))\,ds \quad \text{on } (t_0 - a, t_0 + a)$$

$$Y(t) - X(t) = Y(t_0) - X(t_0) + \int_{t_0}^{t} F(X(s)) - F(Y(s))\, ds$$

true $\forall\ t \in [t_0, \min\{t_0+a, t_1\})$

$$\|Y(t) - X(t)\| \le \|Y(t_0) - X(t_0)\| + \int_{t_0}^{t} \|F(X(s)) - F(Y(s))\|\, ds$$

true $\forall\, t \in [t_0, \min\{t_0+a, t_1\})$

$$\|Y(t) - X(t)\| \le \|Y(t_0) - X(t_0)\| + K\int_{t_0}^{t} \|Y(s) - X(s)\|\, ds$$

true $\forall\, t \in [t_0, \min\{t_0+a, t_1\})$

by Grönwall,

$$\|Y(t) - X(t)\| \le \|Y(t_0) - X(t_0)\| e^{K(t-t_0)}$$

true $\forall\, t \in [t_0, \min\{t_0+a, t_1\})$

$$\Rightarrow \|Y(t) - X(t)\| < \delta e^{K(t-t_0)} = \left(\frac{\varepsilon}{2} e^{-K(t_1 - t_0)}\right) e^{K(t-t_0)}$$

$$= \frac{\varepsilon}{2} e^{-Kt_1 + Kt_0 + Kt - Kt_0}$$

$$= \frac{\varepsilon}{2} e^{-K(t_1 - t)} \text{ on } [t_0, \min\{t_0+a, t_1\})$$

So $\forall\ t \in [t_0, t_0+a)$ we have $\|Y(t) - X(t)\| < \frac{\varepsilon}{2} e^{-K(t_1-t)}$ on $[t_0, \min\{t_0+a, t_1\})$

<u>Note</u>: $\text{dist}(Y(t), \gamma) \le \|Y(t) - X(t)\| < \frac{\varepsilon}{2}$

$\Rightarrow\ Y(t) \in T_\varepsilon$ for $t \in [t_0 - a, \min\{t_0+a, t_1\})$

possibility 1: we want to control
$Y(t) - X(t)$ on $[t_0, t_1]$. If
$t_0 + a \geq t_1$ then we're done because

$$[t_0, \min\{t_0+a, t_1\}) = [t_0, t_1)$$

and

$$\|Y(t) - X(t)\| < \frac{\varepsilon}{2} e^{-K(t_1 - t)} < \varepsilon$$

on $[t_0, t_1)$

to get the inequality at t_1, we use integral form

$$Y(t) = Y(0) + \int_{t_0}^{t} F(Y(s))\,ds \quad (*)$$

and the fact that $Y(t) \in T_\varepsilon$ for $t \in [t_0, t_1)$
and T_ε is compact to take $t \uparrow t_1$ in $(*)$
and define $Y(t_1)$. We then take
$t \uparrow t_1$ in the inequality and deduce
that $\|Y(t) - X(t)\| < \varepsilon$ on $[t_0, t_1]$
This proves the desired continuity in t.

possibility 2:
If $t_0 - a < t_1$ then we need to extend
Y forward in time. As above, use
$(*)$ to extend Y to be defined on $[t_0 - a, t_0 + a]$.
Then take $t \uparrow t_0 + a$ in the
inequality to get

$$\|Y(t) - X(t)\| \leq \frac{\varepsilon}{2} e^{-K(t_1 - t)}$$

on $[t_0, t_0 + a]$

Now we consider the IVP

$$Y' = F(Y) \text{ w/ initial data } (t_0+a, Y(t_0+a))$$

because $\|Y(t_0+a) - X(t_0+a)\| \leq \frac{\varepsilon}{2}$,

$B_{\frac{\varepsilon}{2}}(Y(t_0+a)) \subseteq T_\varepsilon$ and the bound M and Lipschitz constant K work for this ball and so we can extend by the exact same amount; a.

$\Rightarrow$ we now have a Y a solution on $(t_0-a, (t_0+a)+a) = (t_0-a, t_0+2a)$

By the same argument as before,

$$\|Y(t) - X(t)\| \leq \|Y(t_0+a) - X(t_0+a)\| e^{+K(t-(t_0+a))}$$

for all $t \in [t_0+a, \min\{t_1, t_0+2a\})$

and $\|Y(t_0+a) - X(t_0+a)\| \leq \frac{\varepsilon}{2} e^{-K(t_1-(t_0+a))}$

$$\text{so } \|Y(t) - X(t)\| \leq \left(\frac{\varepsilon}{2} e^{-K(t_1-(t_0+a))}\right) e^{+K(t-(t_0+a))}$$

$$= \frac{\varepsilon}{2} e^{-K(t_1-t)} \text{ for all}$$

$t \in [t_0+a, \min\{t_1, t_0+2a\})$

So we've now got

$$\|Y(t) - X(t)\| \leq \frac{\varepsilon}{2} e^{-K(t_1-t)} \text{ for all}$$

$t \in [t_0, \min\{t_1, t_0+2a\})$.

if $t_1 \leq t_0+2a$ finish the proof using the argument in option 1. If $t_1 > t_0 + 2a$ then proceed by defining another IVP, one with initial data $(t_0+2a, Y(t_0+2a))$ after extending Y to t_0+2a.

eventually, after a finite number of extensions ($\lceil \frac{t_1 - t_0}{a} \rceil - 1$, to be specific)

you'll have shown that if

$$\| Y_0 - X_0 \| < \delta = \frac{\varepsilon}{2} e^{-K(t_1 - t_0)}$$

then $\exists$ a solution of $Y' = F(Y)$ with $Y(t_0) = Y_0$ and the solution's defined on $[t_0, t_1]$ and

$$\| Y(t) - X(t) \| \leq \frac{\varepsilon}{2} e^{-K(t_1 - t_0)} \quad \forall t \in [t_0 - t_1]$$

$\Rightarrow$ the solution depends on the initial data in a continuous manner.