Degenerate case ①
$\lambda_1 = \lambda_2$ and only one eigenvector.

Consider $\vec{x}' = \begin{pmatrix} -4 & 1 \\ -4 & 0 \end{pmatrix} \vec{x}$

1) eigenvalues & eigenvectors

$$\det \begin{pmatrix} -4-\lambda & 1 \\ -4 & -\lambda \end{pmatrix} = (-4-\lambda)(-\lambda) + 4 = \lambda^2 + 4\lambda + 4$$
$$= (\lambda+2)^2$$

$\underline{\lambda = -2}$ eigenvectors? Seek $\vec{v} \neq \vec{0}$ so that

$$(A-(-2)I)\vec{v} = \vec{0} \qquad \left(\begin{array}{cc|c} -2 & 1 & 0 \\ -4 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is the only eigenvector.

$$\boxed{\vec{x}_1 = e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

Seek second solution $\vec{x}_2 = e^{-2t}\left[t\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \vec{w}\right]$
where $\vec{w}$ is a solution of $(A-(-2)I)\vec{w} = \vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

There will be infinitely many possible $\vec{w}$ because $\det(A-(-2)I) = 0$. The null space of $A-(-2)I$ is the eigenspace for $\lambda = -2$. It's spanned by the basis $\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$.

$$\left(\begin{array}{cc|c} -2 & 1 & 1 \\ -4 & 2 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} -2 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1/2 & -1/2 \\ 0 & 0 & 0 \end{array}\right)$$

$$\vec{w} = \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad s \in \mathbb{R}$$

Note: I could take some $s_0 \neq 0$

$$\vec{X}_2(t) = e^{-2t}\left[t\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-1/2\\0\end{pmatrix} + s_0\begin{pmatrix}1\\2\end{pmatrix}\right]$$

because of the infinite # of solutions of $(A-(-2)I)\vec{w} = \vec{v}$

$$= e^{-2t}\left[t\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-1/2\\0\end{pmatrix}\right] + s_0 e^{-2t}\begin{pmatrix}1\\2\end{pmatrix}$$

$$= e^{-2t}\left[t\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-1/2\\0\end{pmatrix}\right] + s_0 \vec{X}_1(t)$$

In which case the general solution would be $\vec{X}(t) = c_1\vec{X}_1(t) + c_2\vec{X}_2(t)$

$$= c_1\vec{X}_1(t) + c_2 e^{-2t}\left[t\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-1/2\\0\end{pmatrix}\right] + c_2 s_0 \vec{X}_1(t)$$

$$= \underbrace{(c_1 + c_2 s_0)}\vec{X}_1(t) + c_2 e^{-2t}\left[t\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-1/2\\0\end{pmatrix}\right]$$

look! Including the s_0 term in $\vec{X}_2(t)$ didn't give any extra help in the general solution --- no need for s_0 when we've got c_1! ☺

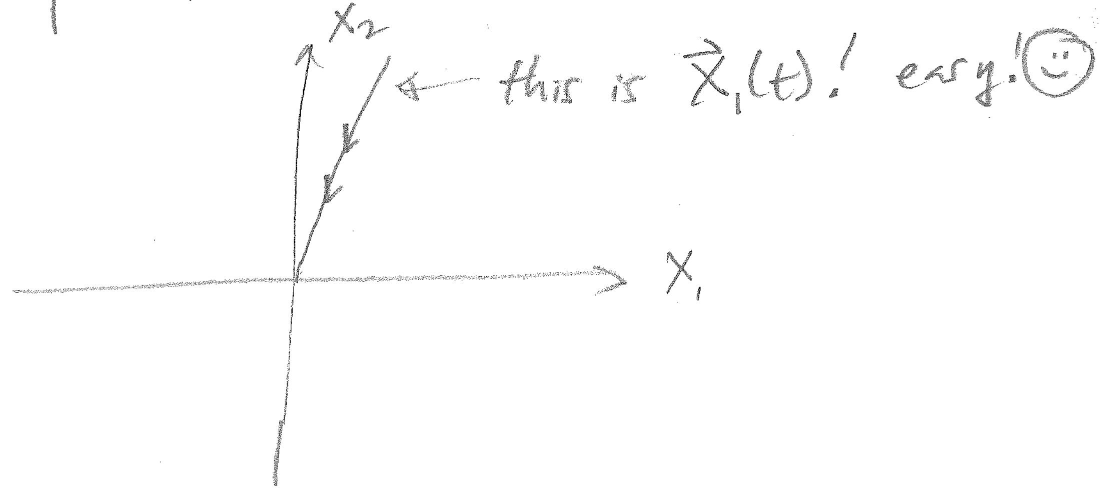
For this reason, when we solve for the generalized eigenvector $\vec{w}$ needed to construct the second solution $\vec{x}_2(t)$ we don't worry about <u>which</u> of the infinitely many possible $\vec{w}$ we choose--- if student X chooses one $\vec{w}$ and student Y chose a different $\vec{w}$ then their resulting $\vec{x}_2(t)$ will agree up to some multiple of $\vec{x}_1(t)$.

Okay. We have

$$\vec{x}_1(t) = e^{-2t}\begin{pmatrix}1\\2\end{pmatrix}$$

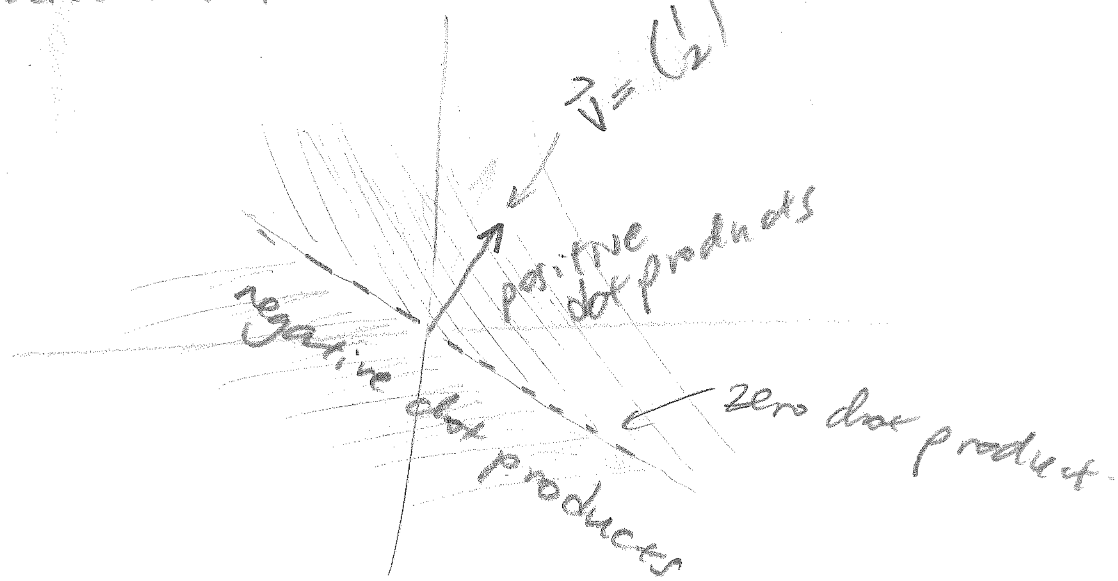
$$\vec{x}_2(t) = e^{-2t}\left[t\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-1/2\\0\end{pmatrix}\right]$$

Now to plot them in the phase plane...



What about plotting $\vec{X}_2(t)$? That's a little harder because its direction changes in time.

Q: Can we understand the direction of $\vec{X}_2(t)$ as a function of time by taking its dot product with $\vec{v}$?



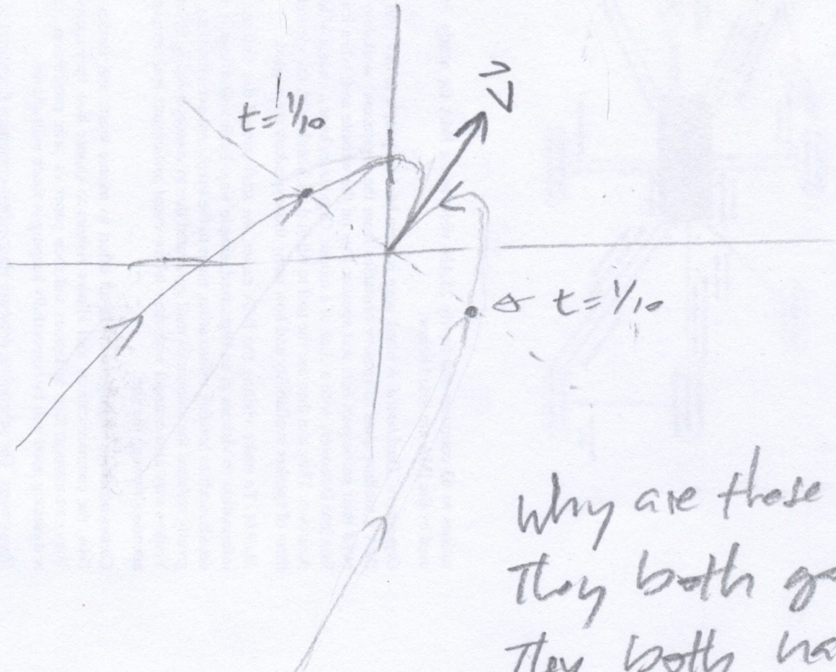
$$\vec{X}_2(t)\cdot\vec{v} = e^{-2t}\left[t\vec{v}+\vec{w}\right]\cdot\vec{v}$$

$$= e^{-2t}\left[t|\vec{v}|^2+\vec{w}\cdot\vec{v}\right]$$

$$= e^{-2t}\left[5t - 1/2\right]$$

So $\vec{X}_2(t)\cdot\vec{v} = \begin{cases} 0 & \text{if } t = 1/10 \\ >0 & \text{if } t > 1/10 \\ <0 & \text{if } t < 1/10 \end{cases}$

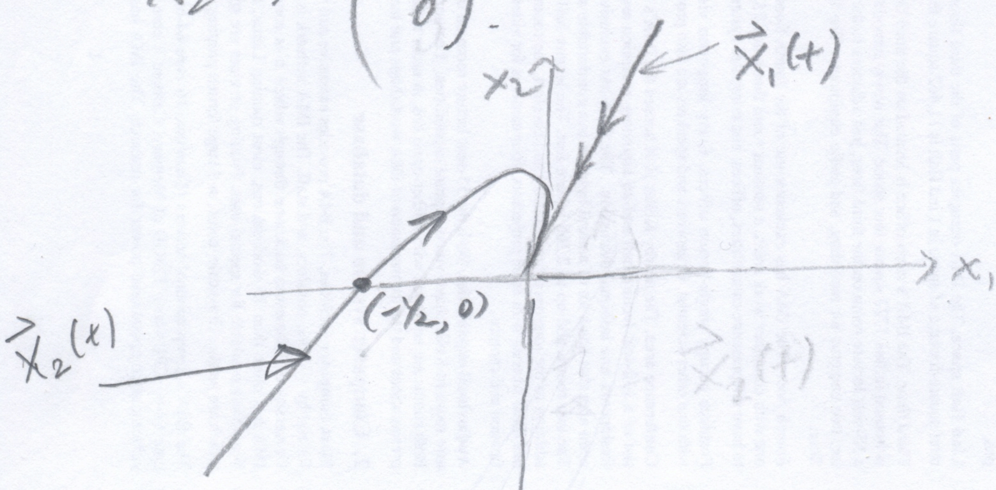
This tells us that we have one of two options for the plot of $\vec{x}_2(t)$:



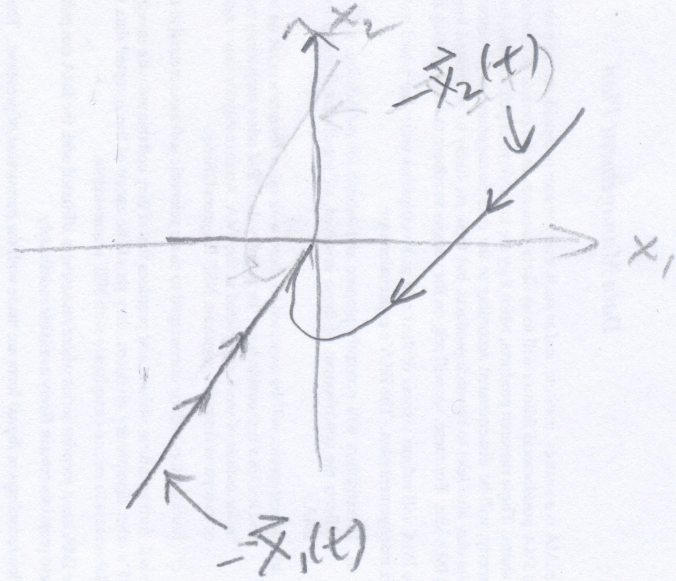
Why are these the 2 options?
They both go to $\vec{0}$ as $t \to \infty$
They both have positive dot product w/ $\vec{v}$ for $t > 1/10$ and negative for $t < 1/10$.

To choose which one is the correct one, evaluate $\vec{x}_2(t)$ at $t=0$

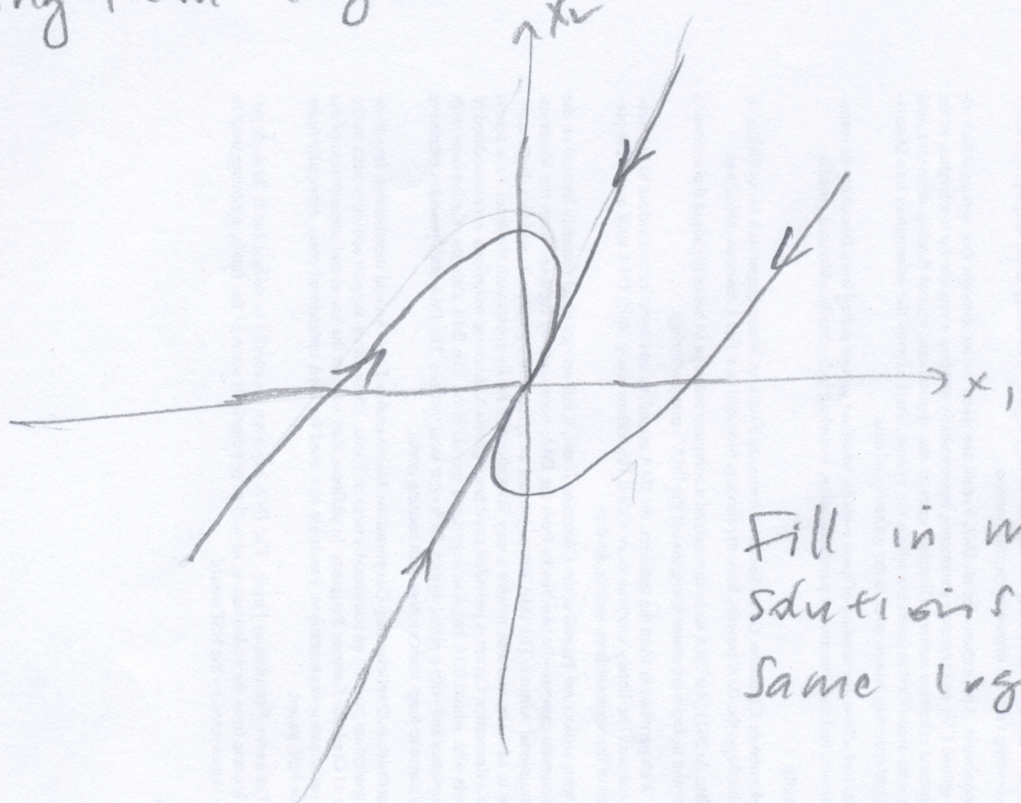
$$\vec{x}_2(0) = \begin{pmatrix} -1/2 \\ 0 \end{pmatrix} \Rightarrow \text{It's the upper of the two!}$$



Because $\vec{x}(t)$ solves $\frac{d\vec{x}}{dt} = A\vec{x}$ implies $-\vec{x}(t)$ is also a solution, we get these two too:



Putting them together:



Fill in more solutions via same logic.

If you don't have time to think and are willing to risk memorizing, you 1 plot $\vec{X}_1(t)$ and 2 choose between the two possible "arms" at the top of 5 by evaluating $\vec{X}_2(t)$ at $t=0$,

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So much for phase plane plots. What about plots of the components of $\vec{X}_2(t)$ versus time?

$$\vec{X}_2(t) = e^{-2t}\left[t\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-1/2\\0\end{pmatrix}\right]$$

$$= e^{-2t}\begin{pmatrix}-1/2+t\\2t\end{pmatrix} = \begin{pmatrix}x_{21}(t)\\x_{22}(t)\end{pmatrix}$$

$$X_{21}(t) = (t-1/2)e^{-2t} \rightarrow X_{21}'(t) = -2(t-1)e^{-2t}$$
$$X_{22}(t) = 2t\,e^{-2t} \rightarrow X_{22}'(t) = (2-4t)e^{-2t}$$

<u>Note:</u>

$X_{21}(t) \downarrow 0$ as $t \to \infty$

$X_{21}(t) \downarrow -\infty$ as $t \to -\infty$

$X_{21}(t) = 0$ at $t = 1/2$

$X_{21}(0) = -1/2 \quad X_{21}'(0) = 2$

$X_{21}'(1) = 0$

$X_{22}(t) \downarrow 0$ as $t \to \infty$

$X_{22}(t) \downarrow -\infty$ as $t \to -\infty$

$X_{22}(t) = 0$ at $t=0$

$X_{22}(0) = 0 \quad X_{22}'(0) = 2$

$X_{22}'(\frac{1}{2}) = 0$

Also, $x_{21}(t) = x_{22}(t)$

$\iff (t - 1/2)e^{-2t} = 2t\,e^{-2t}$

$\iff t - 1/2 = 2t$ $\quad \boxed{t = -1/2}$

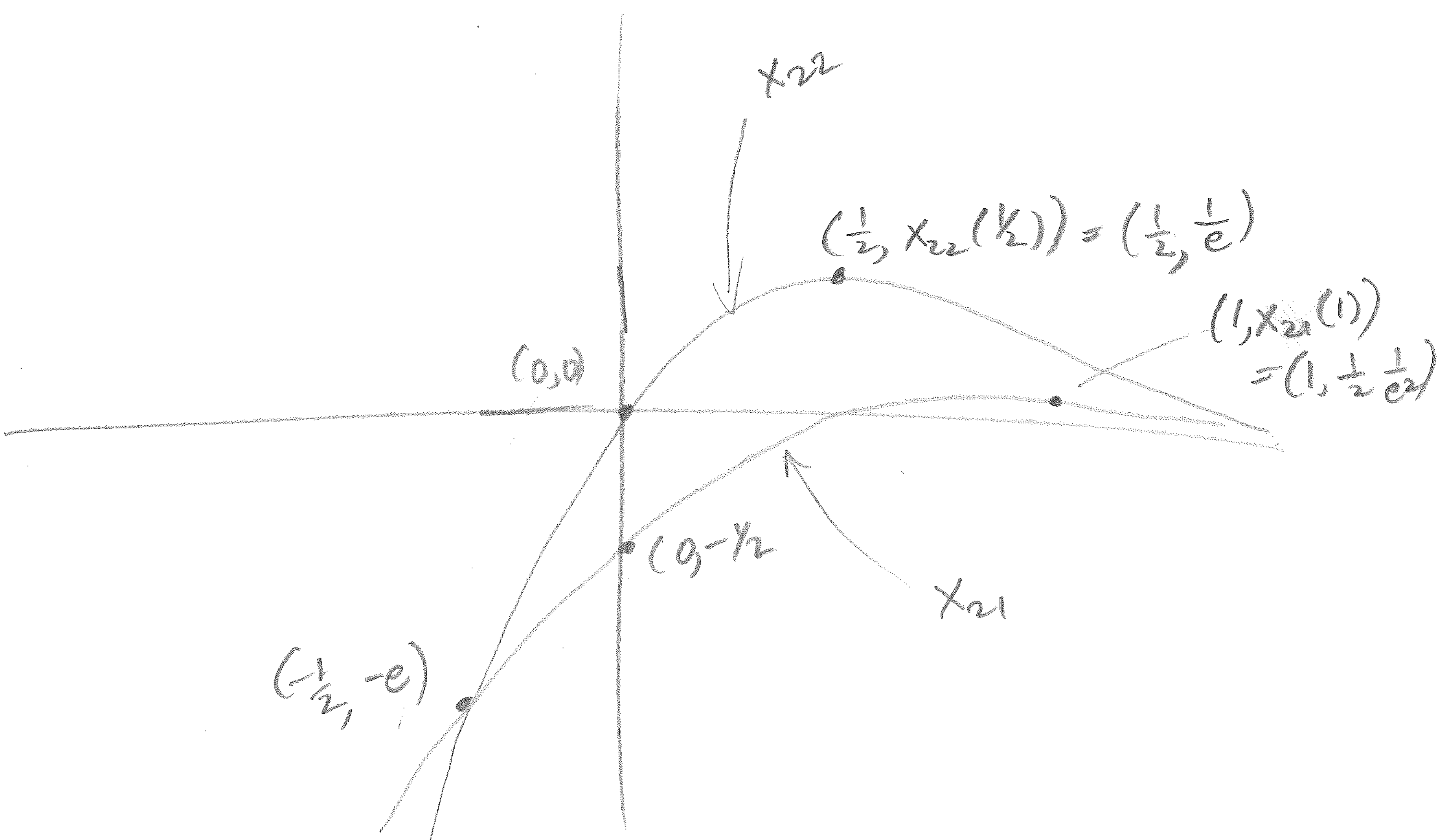
we have $x_{21}(t) < x_{22}(t)$

$\iff (t - 1/2)\,e^{-2t} < 2t\,e^{-2t}$

$\iff (t - 1/2) < 2t$

$\iff t > -1/2.$

Putting this all together



Finally, a quick last comment about the phase plot...

In general,

$$\vec{x}_2(t) = e^{\lambda t}[t\vec{v} + \vec{w}]$$

$$\Rightarrow \vec{x}_2(t) = \begin{pmatrix} e^{\lambda t}(tv_1 + w_1) \\ e^{\lambda t}(tv_2 + w_2) \end{pmatrix} \quad \text{where } \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \ \vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

If $\lambda < 0$ then $\vec{x}_2(t) \to \vec{0}$ as $t \to \infty$
$|\vec{x}_2(t)| \to \infty$ as $t \to -\infty$

and $\dfrac{x_{22}(t)}{x_{21}(t)} = \dfrac{tv_2 + w_2}{tv_1 + w_1} \to \dfrac{v_2}{v_1}$ as $t \to \pm\infty$

So as $t \to +\infty$ $\vec{x}_2(t) \to \vec{0}$ asymptotic to the line $y = \frac{v_2}{v_1}x$ (direction of $\vec{v}$)

And as $t \to -\infty$, $\vec{x}_2(t)$ becomes more and more parallel to $\vec{v}$

If $\lambda > 0$ then it's the same idea but with $+\infty$ and $-\infty$ reversed.