ɔrhood of t_0.
(5) is defined on the whole t-axis, and
ɐble function satisfying the initial con-
al equation (4) for all t. In fact, it was
ı (4) that Napier originally introduced

equation (4), satisfying the condition $\varphi(t_0) =$
ıole interval $a < t < b$ where it is defined.

ollows: Let T be the least upper bound of the
all t, $t_0 \leqslant t < \tau$. By hypothesis, $t_0 \leqslant T \leqslant b$.
cause of the continuity of φ. But then it holds in
epeat the argument leading to (5), replacing t_0
) implies $\varphi(T) > 0$). Thus $T = b$ and formula
$< t \leqslant t_0$ is treated similarly.
ons of (4) with $x_0 > 0$.

1.6 on radioactive decay and growth of bacterial
blem the amount of matter falls off exponentially
bstance decreases to one half the amount initially
he *half-life* of the given substance. In the second
s exponentially with time, and doubles in time
Formula (5) also contains the solution of many

ısity of the atmosphere one half its value at the
erature is constant? (A cubic meter of air weighs

. Elbrus.

scientific journals (both original and review journals).
ulchenko, *Scientometry* (in Russian), Moscow (1969).

Problem 3. Prove that all the solutions of equation (4) satisfying the initial condition $\varphi(t_0) = x_0 < 0$ are also given by formula (5).

It should be noted that none of the functions (5) with $x_0 \neq 0$ vanishes for any value of t. Hence the unique solution of equation (4) such that $x_0 = 0$ is the stationary solution $x \equiv 0$. Thus *formula* (5) *accounts for all the solutions of the differential equation* (4).

In particular, the uniqueness assertion of Theorem 2.3 is valid for equation (4). From this one can easily infer uniqueness for any equation (1) with a differentiable vector field $\mathbf{v}$ and for more general equations as well.

The reason for the failure of uniqueness in the case $\mathbf{v}(x) = x^{2/3}$ is that this field does not fall off fast enough as the point $x = 0$ is approached. Therefore the solution manages to arrive at the singular point in a finite time. An infinite time is required to reach the singular point in the case $\mathbf{v}(x) = kx$, since the integral curves approach each other exponentially. It is characteristic of any differential equation with a differentiable vector field $\mathbf{v}$ that its integral curves do not approach each other more rapidly than exponentially, thereby accounting for the uniqueness. In particular, the uniqueness proof in Theorem 2.3 is easily obtained by comparing the general equation (1) with a suitable equation of the form (4).

2.7. A comparison theorem. Let $\mathbf{v}_1, \mathbf{v}_2$ be real functions continuous on an interval U of the real axis such that $\mathbf{v}_1 < \mathbf{v}_2$, and let φ_1, φ_2 be solutions of the differential equations

$$\dot{x} = \mathbf{v}_1(x), \qquad \dot{x} = \mathbf{v}_2(x) \tag{6}$$

respectively, satisfying the same initial condition $\varphi_1(t_0) = \varphi_2(t_0) = x_0$ (Fig. 20), where φ_1, φ_2 are both defined on the interval $a < t < b$ $(-\infty \leqslant a < b \leqslant +\infty)$.

THEOREM. *The inequality*

$$\varphi_1(t) \leqslant \varphi_2(t) \tag{7}$$

holds for all $t \geqslant t_0$ in the interval (a, b).

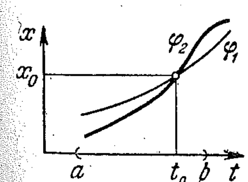

Fig. 20 The slope of φ_2 is greater than that of φ_1 at points with equal x, but not at points with equal t.

prove this!

5. Show how to construct infinitely many solutions, satisfying the initial condition $x(0) = 0$, for the differential equations given in Exercise 1.4 (d) and (f). Can the same be done for the equation in Exercise 1.4 (e) subject to the condition (a) $x(0) = 1$, (b) $x(0) = -1$? Explain your answer.

6. Suppose the differential equation $\dot{x} = X(t, x)$ has the property $X(t, x) = X(-t, -x)$. Prove that if $x = \xi(t)$ is a solution then so is $x = -\xi(-t)$. Find similar results on the symmetry of solutions when: (a) $X(t, x) = -X(-t, x)$; (b) $X(t, x) = -X(t, -x)$. Which of these symmetries appear in Figs. 1.1–1.8?

7. A differential equation of the form

$$\dot{x} = h(t, x) \tag{1}$$

is said to be *homogeneous* if $h(t, x)$ satisfies $h(\alpha t, \alpha x) \equiv h(t, x)$ for all non-zero real α. Show that the isoclines of such an equation are always straight lines through the origin of the t, x-plane.

Use this result to sketch the solution curves of

$$\dot{x} = e^{x/t}, \qquad t \neq 0. \tag{2}$$

Show that the change of variable $x = ut$ allows (1) to be written as a separable equation (Exercise 1.2) for u when $t \neq 0$. Does this result help to obtain the family of solution curves for (2)?

8. Sketch the family of solutions of the differential equation

$$\dot{x} = ax - bx^2, \qquad x > 0, \qquad a \text{ and } b > 0.$$

Obtain the sketch directly from the differential equation itself. Prove that $\dot{x}$ is increasing for $0 < x < a/2b$ and decreasing for $a/2b < x < a/b$. How does this result influence your sketches? How does $\dot{x}$ behave for $a/b < x < \infty$?

9. Show that the substitution $y = x^{-1}$, $x \neq 0$ allows

$$\dot{x} = ax - bx^2$$

to be written as a differential equation for y with the form described in Exercise 1.1. Solve this equation and show that

$$x(t) = ax_0/\{bx_0 + (a - bx_0)\exp(-a(t - t_0))\},$$

where $x(t_0) = x_0$. Verify that the sketches obtained in Exercise 1.8 agree with this result. Can you identify any new qualitative features of the solutions which are not apparent from the original differential equation?

10. Sketch the solution curves of the differential equations:

(a) $\dot{x} = x^2 - t^2 - 1$; (b) $\dot{x} = t - t/x$, $x \neq 0$;
(c) $\dot{x} = (2t + x)/(t - 2x)$, $t \neq 2x$; (d) $\dot{x} = x^2 + t^2$;

by using isoclines and the regions of convexity and concavity.

11. Obtain isoclines and sketch the family of solutions for the following differential equations without finding x as a function of t.

(a) $\dot{x} = x + t$; (b) $\dot{x} = x^3 - x$;
(c) $\dot{x} = xt^2$; (d) $\dot{x} = x \ln x$, $x > 0$;
(e) $\dot{x} = \sinh x$; (f) $\dot{x} = t(x + 1)/(t^2 + 1)$.

What geometrical feature do the isoclines of (b), (d) and (e) have in common? Finally, verify your results using calculus.

Section 1.2

12. Find the fixed points of the following autonomous differential equations:

(a) $\dot{x} = x + 1$; (b) $\dot{x} = x - x^3$; (c) $\dot{x} = \sinh(x^2)$;
(d) $\dot{x} = x^4 - x^3 - 2x^2$; (e) $\dot{x} = x^2 + 1$.

Determine the nature (attractor, repellor or shunt) of each fixed point and hence construct the phase portrait of each equation.

13. Which differential equations, in the following list, have the same phase portrait?

(a) $\dot{x} = \sinh x$; (b) $\dot{x} = ax$, $a > 0$; (c) $\dot{x} = \begin{cases} x \ln|x|, & x \neq 0 \\ 0, & x = 0; \end{cases}$
(d) $\dot{x} = \sin x$; (e) $\dot{x} = x^3 - x$; (f) $\dot{x} = \tanh x$.

Explain, in your own words, the significance of two differential equations having the same phase portrait.

14. Consider the parameter dependent differential equation

$$\dot{x} = (x - \lambda)(x^2 - \lambda), \qquad \lambda \text{ real}.$$

Find all possible phase portraits that could occur for this equation together with the intervals of λ in which they occur.

15. How many distinct qualitative types of phase portrait can occur on the phase line for a differential equation with three fixed points? What is the formula for the number of distinct phase portraits in the general case with n fixed points?

16. Show that the phase portrait of

$$\dot{x} = (a-x)(b-x)$$

is qualitatively the same as that of

$$\dot{y} = y(y-c)$$

for all real a, b, c; $a \neq b$, $c \neq 0$. Show, however, that a transformation, $y = kx + l$, which takes the first equation into the second, exists if and only if $c = b - a$ or $a - b$.

17. Consider the differential equation

$$\dot{x} = x^3 + ax - b.$$

Show that there is a curve C in the a, b-plane which separates this plane into two regions A and B such that: if $(a, b) \in A$ the phase portrait consists of a single repellor and if $(a, b) \in B$ it has two repellors separated by an attractor. Let $a < 0$ be fixed; describe the change in configuration of the fixed points as b varies from $-\infty$ to ∞.

18. A substance γ is formed in a chemical reaction between substances α and β. In the reaction each gram of γ is produced by the combination of p grams of α and $q = 1 - p$ grams of β. The rate of formation of γ at any instant of time, t, is equal to the product of the masses of α and β that remain uncombined at that instant. If a grams of α and b grams of β are brought together at $t = 0$, show that the differential equation governing the mass, $x(t)$, of γ present at time $t > 0$ is

$$\dot{x} = (a - px)(b - qx).$$

Assume $a/p > b/q$ and construct the phase portrait for this equation. What is the maximum amount of γ that can possibly be produced in this experiment?

Section 1.3

19. Find the fixed points of the following systems of differential equations in the plane:

(a) $\dot{x}_1 = x_1(a - bx_2)$, $\dot{x}_2 = -x_2(c - dx_1)$, $a, b, c, d > 0$;

(b) $\dot{x}_1 = x_2$, $\dot{x}_2 = -\sin x_1$;

(c) $\dot{x}_1 = x_2$, $\dot{x}_2 = x_2(1 - x_1^2) - x_1$;

(d) $\dot{x}_1 = x_1(2 - x_1 - 2x_2)$, $\dot{x}_2 = x_2(2 - 2x_1 - x_2)$;

(e) $\dot{x}_1 = \sin x_1$, $\dot{x}_2 = \cos x_2$.

✱ **20.** Sketch the following parametrized families of curves in the plane:

(a) $(x_1, x_2) = (a\cos t, a\sin t)$; (b) $(x_1, x_2) = (a\cos t, 2a\sin t)$;
(c) $(x_1, x_2) = (ae^t, be^{-2t})$; (d) $(x_1, x_2) = (ae^t + be^{-t}, ae^t - be^{-t})$;
(e) $(x_1, x_2) = (ae^t + be^{2t}, be^{2t})$;

where $a, b \in \mathbb{R}$. Find the systems of differential equations in the plane for which these curves form the phase portrait.

21. Use the sketches obtained in Exercise 1.20, to arrange the families of curves (a)–(e) in groups with the same type of fixed point at the origin of the x_1x_2-plane.

22. Consider the phase portrait of the system

$$\dot{x}_1 = X_1(x_1, x_2), \qquad \dot{x}_2 = X_2(x_1, x_2). \tag{1}$$

Show that the system

$$\dot{y}_1 = -X_1(y_1, y_2), \qquad \dot{y}_2 = -X_2(y_1, y_2)$$

has trajectories with the same shape but with the reverse orientation to those of (1). Verify your result by obtaining solutions for:

(a) $\dot{x}_1 = x_1$, $\dot{x}_2 = x_2$; (b) $\dot{x}_1 = x_1$, $\dot{x}_2 = 2x_2$

and comparing with Figs. 1.23 and 1.24.

Section 1.4

✱ **23.** Use the change of variable

$$x_1 = y_1 + y_2, \qquad x_2 = y_1 - y_2$$

to decouple the pair of differential equations

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1.$$

Solving for I, we get

$$I = \int e^{kt}\sin(\omega t)\,dt = \frac{1}{k^2+\omega^2}e^{kt}\,[k\sin(\omega t) - \omega\cos(\omega t)]. \tag{45}$$

Of course, the integral in Eq. (44) can also be easily done using a computer algebra system.
Substituting the result (45) into Eq. (44) then gives

$$u = T_0 + \frac{kA}{k^2+\omega^2}\,[k\sin(\omega t) - \omega\cos(\omega t)] + ce^{-kt},$$

in agreement with Eq. (11) of Section 1.2. Some graphs of the solution are shown in Figure 1.2.8.

PROBLEMS

In each of Problems 1 through 12:
(a) Draw a direction field for the given differential equation.
(b) Based on an inspection of the direction field, describe how solutions behave for large t.
(c) Find the general solution of the given differential equation, and use it to determine how solutions behave as $t \to \infty$.

1. $y' + 4y = t + e^{-2t}$
2. $y' - 2y = t^2e^{2t}$
3. $y' + y = te^{-t} + 1$
4. $y' + (1/t)y = 5\cos 2t, \quad t > 0$
5. $y' - 2y = 3e^t$
6. $ty' + 2y = \sin t, \quad t > 0$
7. $y' + 2ty = 16te^{-t^2}$
8. $(1 + t^2)y' + 4ty = (1 + t^2)^{-2}$
9. $2y' + y = 3t$
10. $ty' - y = t^3e^{-t}, \quad t > 0$
11. $y' + y = 5\sin 2t$
12. $2y' + y = 3t^2$

In each of Problems 13 through 20, find the solution of the given initial value problem.

13. $y' - y = 2te^{2t}, \quad y(0) = 1$
14. $y' + 2y = te^{-2t}, \quad y(1) = 0$
15. $ty' + 4y = t^2 - t + 1, \quad y(1) = \frac{1}{4}, \quad t > 0$
16. $y' + (2/t)y = (\cos t)/t^2, \quad y(\pi) = 0, \quad t > 0$
17. $y' - 2y = e^{2t}, \quad y(0) = 2$
18. $ty' + 2y = \sin t, \quad y(\pi/2) = 3, \quad t > 0$
19. $t^3y' + 4t^2y = e^{-t}, \quad y(-1) = 0, \quad t < 0$
20. $ty' + (t+1)y = t, \quad y(\ln 2) = 1, \quad t > 0$

In each of Problems 21 through 23:
(a) Draw a direction field for the given differential equation. How do solutions appear to behave as t becomes large? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.
(b) Solve the initial value problem and find the critical value a_0 exactly.
(c) Describe the behavior of the solution corresponding to the initial value a_0.

21. $y' - \frac{1}{3}y = 3\cos t, \quad y(0) = a$
22. $2y' - y = e^{t/3}, \quad y(0) = a$
23. $3y' - 2y = e^{-\pi t/2}, \quad y(0) = a$

In each of Problems 24 through 26:
(a) Draw a direction field for the given differential equation. How do solutions appear to behave as $t \to 0$? Does the behavior depend on the choice of the initial value a? Let a_0 be the value of a for which the transition from one type of behavior to another occurs. Estimate the value of a_0.
(b) Solve the initial value problem and find the critical value a_0 exactly.
(c) Describe the behavior of the solution corresponding to the initial value a_0.

24. $ty' + (t+1)y = 2te^{-t}, \quad y(1) = a, \quad t > 0$
25. $ty' + 2y = (\sin t)/t, \quad y(-\pi/2) = a, \quad t < 0$
26. $(\sin t)y' + (\cos t)y = e^t, \quad y(1) = a, \quad 0 < t < \pi$
27. Consider the initial value problem
$$y' + \tfrac{1}{2}y = 2\cos t, \quad y(0) = -1.$$
Find the coordinates of the first local maximum point of the solution for $t > 0$.
28. Consider the initial value problem
$$y' + \tfrac{4}{3}y = 1 - \tfrac{1}{4}t, \quad y(0) = y_0.$$
Find the value of y_0 for which the solution touches, but does not cross, the t-axis.
29. Consider the initial value problem
$$y' + \tfrac{1}{4}y = 3 + 2\cos 2t, \quad y(0) = 0.$$

(a) Find the solution of this initial value problem and describe its behavior for large t.
(b) Determine the value of t for which the solution first intersects the line $y = 12$.

30. Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 3\sin t, \qquad y(0) = y_0$$

remains finite as $t \to \infty$.

31. Consider the initial value problem

$$y' - \tfrac{3}{2}y = 3t + 2e^t, \qquad y(0) = y_0.$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

32. Show that all solutions of $2y' + ty = 2$ [Eq. (36) of the text] approach a limit as $t \to \infty$, and find the limiting value.
Hint: Consider the general solution, Eq. (42), and use L'Hôpital's rule on the first term.

33. Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$.
Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

In each of Problems 34 through 37, construct a first order linear differential equation whose solutions have the required behavior as $t \to \infty$. Then solve your equation and confirm that the solutions do indeed have the specified property.

34. All solutions have the limit 3 as $t \to \infty$.

35. All solutions are asymptotic to the line $y = 4 - t$ as $t \to \infty$.

36. All solutions are asymptotic to the line $y = 2t - 5$ as $t \to \infty$.

37. All solutions approach the curve $y = 2 - t^2$ as $t \to \infty$.

38. Consider the initial value problem

$$y' + ay = g(t), \qquad y(t_0) = y_0.$$

Assume that a is a positive constant and that $g(t) \to g_0$ as $t \to \infty$. Show that $y(t) \to g_0/a$ as $t \to \infty$. Construct an example with a nonconstant $g(t)$ that illustrates this result.

39. **Variation of Parameters.** Consider the following method of solving the general linear equation of first order:

$$y' + p(t)y = g(t). \qquad \text{(i)}$$

(a) If $g(t) = 0$ for all t, show that the solution is

$$y = A\exp\left[-\int p(t)\,dt\right], \qquad \text{(ii)}$$

where A is a constant.
(b) If $g(t)$ is not everywhere zero, assume that the solution of Eq. (i) is of the form

$$y = A(t)\exp\left[-\int p(t)\,dt\right], \qquad \text{(iii)}$$

where A is now a function of t. By substituting for y in the given differential equation, show that $A(t)$ must satisfy the condition

$$A'(t) = g(t)\exp\left[\int p(t)\,dt\right]. \qquad \text{(iv)}$$

(c) Find $A(t)$ from Eq. (iv). Then substitute for $A(t)$ in Eq. (iii) and determine y. Verify that the solution obtained in this manner agrees with that of Eq. (28) in the text. This technique is known as the method of **variation of parameters**; it is discussed in detail in Section 4.7 in connection with second order linear equations.

In each of Problems 40 through 43 use the method of Problem 39 to solve the given differential equation.

40. $y' - 6y = t^6 e^{6t}$

41. $y' + (1/t)y = 3\cos 2t, \quad t > 0$

42. $ty' + 2y = \sin t, \quad t > 0$

43. $2y' + y = 3t^2$

2.3 Modeling with First Order Equations

Differential equations are of interest to nonmathematicians primarily because of the possibility of using them to investigate a wide variety of problems in engineering and in the physical, biological, and social sciences. One reason for this is that mathematical models and their solutions lead to equations relating the variables and parameters in the problem. These equations often enable you to make predictions about how the natural process will behave in various circumstances. For example, all the figures in Section 2.2 show solution features that can be found by examining the parameter dependence of solution formulas. These features can be interpreted in terms of the physical behavior of the systems that the differential equations model. Furthermore, it is often easy to vary parameters in the mathematical model over wide ranges, whereas this may be very time-consuming or expensive, if likely that the singularities will depend on the initial condition as well as the differential equation.

PROBLEMS

Existence and Uniqueness of Solutions. In each of Problems 1 through 6, use Theorem 2.4.1 to determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist.

1. $(t-3)y' + (\ln t)y = 2t, \qquad y(1) = 2$
2. $t(t-4)y' + y = 0, \qquad y(2) = 1$
3. $y' + (\tan t)y = \sin t, \qquad y(\pi) = 0$
4. $(4-t^2)y' + 2ty = 3t^2, \qquad y(-3) = 1$
5. $(4-t^2)y' + 2ty = 3t^2, \qquad y(1) = -3$
6. $(\ln t)y' + y = \cot t, \qquad y(2) = 3$

In each of Problems 7 through 12, state where in the ty-plane the hypotheses of Theorem 2.4.2 are satisfied.

7. $y' = \dfrac{t-y}{2t+5y}$
8. $y' = (1-t^2-y^2)^{1/2}$
9. $y' = \dfrac{\ln|ty|}{1-t^2+y^2}$
10. $y' = (t^2+y^2)^{3/2}$
11. $\dfrac{dy}{dt} = \dfrac{1+t^2}{3y-y^2}$
12. $\dfrac{dy}{dt} = \dfrac{(\cot t)y}{1+y}$

13. Consider the initial value problem $y' = y^{1/3}$, $y(0) = 0$ from Example 3 in the text.

(a) Is there a solution that passes through the point (1, 1)? If so, find it.
(b) Is there a solution that passes through the point (2, 1)? If so, find it.
(c) Consider all possible solutions of the given initial value problem. Determine the set of values that these solutions attain at $t = 2$.

14. (a) Verify that both $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are solutions of the initial value problem

$$y' = \frac{-t + (t^2+4y)^{1/2}}{2}, \qquad y(2) = -1.$$

Where are these solutions valid?
(b) Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of Theorem 2.4.2.
(c) Show that $y = ct + c^2$, where c is an arbitrary constant, satisfies the differential equation in part (a) for $t \geq -2c$. If $c = -1$, the initial condition is also satisfied, and the solution $y = y_1(t)$ is obtained. Show that there is no choice of c that gives the second solution $y = y_2(t)$.

Dependence of Solutions on Initial Conditions. In each of Problems 15 through 18, solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

15. $y' = -4t/y, \qquad y(0) = y_0$
16. $y' = 2ty^2, \qquad y(0) = y_0$
17. $y' + y^3 = 0, \qquad y(0) = y_0$
18. $y' = t^2/y(1+t^3), \qquad y(0) = y_0$

In each of Problems 19 through 22, draw a direction field and plot (or sketch) several solutions of the given differential equation. Describe how solutions appear to behave as t increases and how their behavior depends on the initial value y_0 when $t = 0$.

19. $y' = ty(3-y)$
20. $y' = y(3-ty)$
21. $y' = -y(3-ty)$
22. $y' = t - 1 - y^2$

Linearity Properties

23. (a) Show that $\phi(t) = e^{2t}$ is a solution of $y' - 2y = 0$ and that $y = c\phi(t)$ is also a solution of this equation for any value of the constant c.
(b) Show that $\phi(t) = 1/t$ is a solution of $y' + y^2 = 0$ for $t > 0$ but that $y = c\phi(t)$ is not a solution of this equation unless $c = 0$ or $c = 1$. Note that the equation of part (b) is nonlinear, whereas that of part (a) is linear.

24. Show that if $y = \phi(t)$ is a solution of $y' + p(t)y = 0$, then $y = c\phi(t)$ is also a solution for any value of the constant c.

25. Let $y = y_1(t)$ be a solution of

$$y' + p(t)y = 0, \qquad \text{(i)}$$

and let $y = y_2(t)$ be a solution of

$$y' + p(t)y = g(t). \qquad \text{(ii)}$$

Show that $y = y_1(t) + y_2(t)$ is also a solution of Eq. (ii).

26. (a) Show that the solution (7) of the general linear equation (1) can be written in the form

$$y = cy_1(t) + y_2(t), \qquad \text{(i)}$$

where c is an arbitrary constant. Identify the functions y_1 and y_2.
(b) Show that y_1 is a solution of the differential equation

$$y' + p(t)y = 0, \qquad \text{(ii)}$$

corresponding to $g(t) = 0$.

(c) Show that y_2 is a solution of the full linear equation (1). We see later (e.g., in Section 4.5) that solutions of higher order linear equations have a pattern similar to Eq. (i).

Discontinuous Coefficients. Linear differential equations sometimes occur in which one or both of the functions p and g have jump discontinuities. If t_0 is such a point of discontinuity, then it is necessary to solve the equation separately for $t < t_0$ and $t > t_0$. Afterward, the two solutions are matched so that y is continuous at t_0. This is accomplished by a proper choice of the arbitrary constants. Problems 27 and 28 illustrate this situation. Note in each case that it is impossible to make y' continuous at t_0: explain why, just from examining the differential equations.

27. Solve the initial value problem

$$y' + 2y = g(t), \qquad y(0) = 0,$$

where

$$g(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & t > 1. \end{cases}$$

28. Solve the initial value problem

$$y' + p(t)y = 0, \qquad y(0) = 1,$$

where

$$p(t) = \begin{cases} 2, & 0 \le t \le 1, \\ 1, & t > 1. \end{cases}$$

29. Consider the initial value problem

$$y' + p(t)y = g(t), \qquad y(t_0) = y_0. \tag{i}$$

(a) Show that the solution of the initial value problem (i) can be written in the form

$$y = y_0 \exp\left(-\int_{t_0}^{t} p(s)\, ds\right) + \int_{t_0}^{t} \exp\left(-\int_{s}^{t} p(r)\, dr\right) g(s)\, ds.$$

(b) Assume that $p(t) \ge p_0 > 0$ for all $t \ge t_0$ and that $g(t)$ is bounded for $t \ge t_0$ (i.e., there is a constant M such that $|g(t)| \le M$ for all $t \ge t_0$). Show that the solution of the initial value problem (i) is bounded for $t \ge t_0$.

(c) Construct an example with nonconstant $p(t)$ and $g(t)$ that illustrates this result.

2.5 Autonomous Equations and Population Dynamics

In Section 1.2 we first encountered the following important class of first order equations in which the independent variable does not appear explicitly.

DEFINITION 2.5.1

Autonomous Equation. A differential equation that can be written as

$$\frac{dy}{dt} = f(y). \tag{1}$$

is said to be **autonomous.**

We will now discuss these equations in the context of the growth or decline of the population of a given species, an important issue in fields ranging from medicine to ecology to global economics. A number of other applications are mentioned in some of the problems. Recall that in Section 2.1 we considered the special case of Eq. (1) in which the form of the right side is $f(y) = ay + b$.

Equation (1) is separable, and it can be solved using the approach discussed in Section 2.1. However, the main purpose of this section is to show how geometrical methods can be used to obtain important qualitative information about solutions directly from the differential equation, without solving the equation. Of fundamental importance in this effort are the concepts of stability and instability of solutions of differential equations. These ideas were possible to show that the curve of minimum time is given by a function $y = \phi(x)$ that satisfies the differential equation

$$(1 + y'^2)y = k^2, \qquad \text{(i)}$$

where k^2 is a certain positive constant to be determined later.

(a) Solve Eq. (i) for y'. Why is it necessary to choose the positive square root?

(b) Introduce the new variable t by the relation

$$y = k^2 \sin^2 t. \qquad \text{(ii)}$$

Show that the equation found in part (a) then takes the form

$$2k^2 \sin^2 t\, dt = dx. \qquad \text{(iii)}$$

(c) Letting $\theta = 2t$, show that the solution of Eq. (iii) for which $x = 0$ when $y = 0$ is given by

$$x = k^2(\theta - \sin\theta)/2, \quad y = k^2(1 - \cos\theta)/2. \qquad \text{(iv)}$$

Equations (iv) are parametric equations of the solution of Eq. (i) that passes through (0, 0). The graph of Eqs. (iv) is called a **cycloid**.

(d) If we make a proper choice of the constant k, then the cycloid also passes through the point (x_0, y_0) and is the solution of the brachistochrone problem. Find k if $x_0 = 1$ and $y_0 = 2$.

2.4 Differences Between Linear and Nonlinear Equations

Up to now, we have been primarily concerned with showing that first order differential equations can be used to investigate many different kinds of problems in the natural sciences, and with presenting methods of solving such equations if they are either linear or separable. Now it is time to turn our attention to some more general questions about differential equations and to explore, in more detail, some important ways in which nonlinear equations differ from linear ones.

Existence and Uniqueness of Solutions. So far, we have discussed a number of initial value problems, each of which had a solution and apparently only one solution. This raises the question of whether this is true of all initial value problems for first order equations. In other words, does every initial value problem have exactly one solution? This may be an important question even for nonmathematicians. If you encounter an initial value problem in the course of investigating some physical problem, you might want to know that it has a solution before spending very much time and effort in trying to find it. Further, if you are successful in finding one solution, you might be interested in knowing whether you should continue a search for other possible solutions or whether you can be sure that there are no other solutions. For linear equations, the answers to these questions are given by the following fundamental theorem.

THEOREM 2.4.1

If the functions p and g are continuous on an open interval $I = (\alpha, \beta)$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t) \qquad (1)$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0, \qquad (2)$$

where y_0 is an arbitrary prescribed initial value.

Observe that Theorem 2.4.1 states that the given initial value problem *has* a solution and also that the problem has *only one* solution. In other words, the theorem asserts both the *existence* and *uniqueness* of the solution of the initial value problem (1), (2).

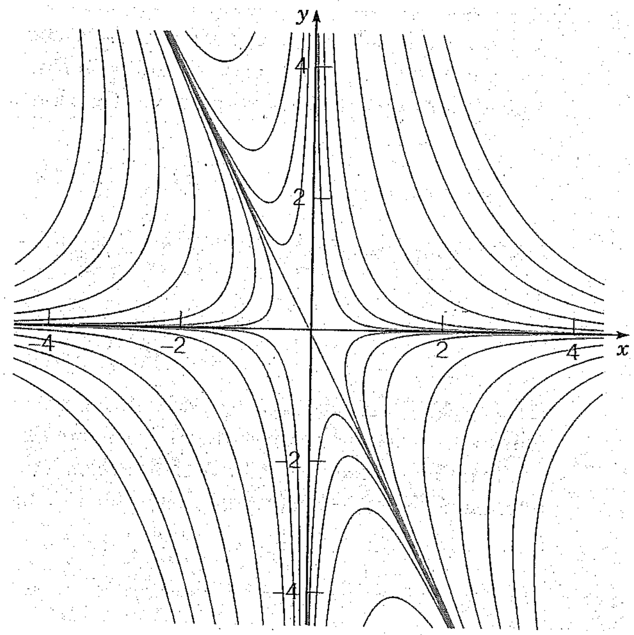

FIGURE 2.6.3 Integral curves of Eq. (19).

You may also verify that a second integrating factor of Eq. (19) is

$$\mu(x, y) = \frac{1}{xy(2x + y)},$$

and that the same solution is obtained, though with much greater difficulty, if this integrating factor is used (see Problem 32).

PROBLEMS

Exact Equations. In each of Problems 1 through 12:
(a) Determine whether the equation is exact. If it is exact, then:
(b) Solve the equation.
(c) Use a computer to draw several integral curves.

1. $(2x + 3) + (2y - 2)y' = 0$
2. $(2x + 4y) + (2x - 2y)y' = 0$
3. $(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$
4. $(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$
5. $\dfrac{dy}{dx} = -\dfrac{4x + 2y}{2x + 3y}$
6. $\dfrac{dy}{dx} = -\dfrac{4x - 2y}{2x - 3y}$
7. $(e^x \sin y - 2y \sin x) + (e^x \cos y + 2 \cos x)y' = 0$
8. $(e^x \sin y + 3y) - (3x - e^x \sin y)y' = 0$
9. $(ye^{xy} \cos 2x - 2e^{xy} \sin 2x + 2x) + (xe^{xy} \cos 2x - 3)y' = 0$
10. $(y/x + 6x) + (\ln x - 2)y' = 0, \quad x > 0$
11. $(x \ln y + xy) + (y \ln x + xy)y' = 0; \quad x > 0, \quad y > 0$
12. $\dfrac{x}{(x^2 + y^2)^{3/2}} + \dfrac{y}{(x^2 + y^2)^{3/2}}y' = 0$

In each of Problems 13 and 14, solve the given initial value problem and determine, at least approximately, where the solution is valid.

13. $(2x - y) + (2y - x)y' = 0, \quad y(1) = 3$
14. $(9x^2 + y - 1) - (4y - x)y' = 0, \quad y(1) = 0$

In each of Problems 15 and 16, find the value of b for which the given equation is exact, and then solve it using that value of b.

15. $(xy^2 + bx^2y) + (x + y)x^2y' = 0$
16. $(ye^{2xy} + x) + bxe^{2xy}y' = 0$
17. Assume that Eq. (6) meets the requirements of Theorem 2.6.1 in a rectangle R and is therefore exact. Show that a

possible function $\psi(x, y)$ is

$$\psi(x, y) = \int_{x_0}^{x} M(s, y_0)\, ds + \int_{y_0}^{y} N(x, t)\, dt,$$

where (x_0, y_0) is a point in R.

18. Show that any separable equation

$$M(x) + N(y)y' = 0$$

is also exact.

Integrating Factors. In each of Problems 19 through 22:

(a) Show that the given equation is not exact but becomes exact when multiplied by the given integrating factor.
(b) Solve the equation.
(c) Use a computer to draw several integral curves.

19. $x^2y^3 + x(1 + y^2)y' = 0, \qquad \mu(x, y) = 1/xy^3$

20. $\left(\dfrac{\sin y}{y} - 2e^{-x}\sin x\right) + \left(\dfrac{\cos y + 2e^{-x}\cos x}{y}\right)y' = 0, \qquad \mu(x, y) = ye^x$

21. $y + (2x - ye^y)y' = 0, \qquad \mu(x, y) = y$

22. $(x + 2)\sin y + x\cos y y' = 0, \qquad \mu(x, y) = xe^x$

23. Show that if $(N_x - M_y)/M = Q$, where Q is a function of y only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form

$$\mu(y) = \exp \int Q(y)\, dy.$$

24. Show that if $(N_x - M_y)/(xM - yN) = R$, where R depends on the quantity xy only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form $\mu(xy)$. Find a general formula for this integrating factor.

In each of Problems 25 through 31:
(a) Find an integrating factor and solve the given equation.
(b) Use a computer to draw several integral curves.

25. $(3x^2y + 2xy + y^3) + (x^2 + y^2)y' = 0$

26. $y' = e^{2x} + y - 1$

27. $1 + (x/y - \sin y)y' = 0$

28. $y + (2xy - e^{-2y})y' = 0$

29. $e^x + (e^x \cot y + 2y \csc y)y' = 0$

30. $4\left(\dfrac{x^3}{y^2} + \dfrac{3}{y}\right) + 3\left(\dfrac{x}{y^2} + 4y\right)y' = 0$

31. $\left(3x + \dfrac{6}{y}\right) + \left(\dfrac{x^2}{y} + 3\dfrac{y}{x}\right)\dfrac{dy}{dx} = 0$

Hint: See Problem 24.

32. Use the integrating factor $\mu(x, y) = [xy(2x+y)]^{-1}$ to solve the differential equation

$$(3xy + y^2) + (x^2 + xy)y' = 0.$$

Verify that the solution is the same as that obtained in Example 4 with a different integrating factor.

2.7 Substitution Methods

In the preceding sections we developed techniques for solving three important classes of differential equations, namely, separable, linear, and exact. But the differential equations arising in many, if not most, applications do not fall into these three categories. In some cases, though, an appropriate substitution or a change of variable can be used to transform the equation into a member of one of these classes. This section focuses on two such types of equations.

Homogeneous Differential Equations

A function $f(x, y)$ is **homogeneous of degree k** if

$$f(\lambda x, \lambda y) = \lambda^k f(x, y), \tag{1}$$

for all (x, y) in its domain. For example, $f(x, y) = \frac{x^2 - xy + y^2}{xy}$ is homogeneous of degree 0 because

$$f(\lambda x, \lambda y) = \frac{\lambda^2x^2 - (\lambda x)(\lambda y) + \lambda^2y^2}{(\lambda x)(\lambda y)} = \frac{\lambda^2\left[x^2 - xy + y^2\right]}{\lambda^2\,[xy]} = \lambda^0 f(x, y)$$

FIGURE 2.7.3 Phase line for Eq. (29).

Relationships Among Classes of Equations

We have developed techniques for solving separable, linear, and exact equations, as well as transformation methods used to convert other equations (e.g., homogeneous and Bernoulli equations) into one of these types. The initial struggle you face when solving a first order differential equation is determining which of these techniques, if any, is applicable. In fact, sometimes more than one approach can be used to solve an equation.

The interrelationships among the main equation types are displayed in Figure 2.7.4. We use arrows to indicate that the type of equation listed near its tail can be transformed into the type of equation to which the arrowhead points.

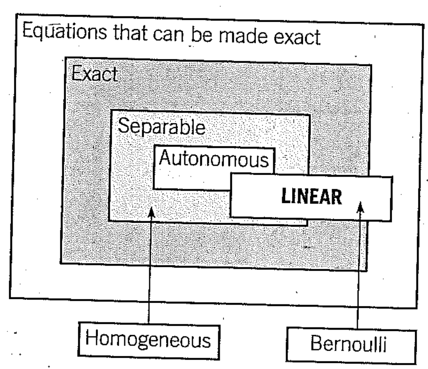

FIGURE 2.7.4 Interrelationships among equation types.

There is a collection of exercises at the end of the section that will challenge you to classify equations and to solve those that have multiple classifications using more than one method.

PROBLEMS

Homogeneous Differential Equations. In each of Problems 1 through 10:
(a) Determine if the equation is homogeneous. If it is homogeneous, then:
(b) Solve the equation.
(c) Use a computer to draw several integral curves.

1. $y\dfrac{dy}{dx} = x + 1$

2. $(y^4 + 1)\dfrac{dy}{dx} = x^4 + 1$

3. $\dfrac{3x^3 - xy^2}{3x^2y + y^3} \cdot \dfrac{dy}{dx} = 1$

4. $x(x-1)\dfrac{dy}{dx} = y(y+1)$

5. $\sqrt{x^2 - y^2} + y = x\dfrac{dy}{dx}$

6. $xy\dfrac{dy}{dx} = (x+y)^2$

7. $\dfrac{dy}{dx} = \dfrac{4y - 7x}{5x - y}$

8. $x\dfrac{dy}{dx} - 4\sqrt{y^2 - x^2} = y, \qquad y > 0$

9. $\dfrac{dy}{dx} = \dfrac{y^4 + 2xy^3 - 3x^2y^2 - 2x^3y}{2x^2y^2 - 2x^3y - 2x^4}$

10. $\left(y + xe^{x/y}\right)\dfrac{dy}{dx} = ye^{x/y}$

In Problems 11 and 12, solve the given initial value problem and determine, at least approximately, where the solution is valid.

11. $xy\dfrac{dy}{dx} = x^2 + y^2, \quad y(2) = 1$

12. $\dfrac{dy}{dx} = \dfrac{x + y}{x - y}, \quad y(5) = 8$

Bernoulli Differential Equations. In each of Problems 13 through 22:

(a) Write the Bernoulli equation in the proper form (19).
(b) Solve the equation.
(c) Use a computer to draw several integral curves.

13. $t\dfrac{dy}{dt} + y = t^2y^2$

14. $\dfrac{dy}{dt} = y\left(ty^3 - 1\right)$

15. $\dfrac{dy}{dt} + \dfrac{3}{t}y = t^2y^2$

16. $t^2y' + 2ty - y^3 = 0, \qquad t > 0$

17. $5(1 + t^2)\dfrac{dy}{dt} = 4ty\left(y^3 - 1\right)$

18. $3t\dfrac{dy}{dt} + 9y = 2ty^{5/3}$

19. $\dfrac{dy}{dt} = y + \sqrt{y}$

20. $y' = ry - ky^2$, $r > 0$ and $k > 0$. This equation is important in population dynamics and is discussed in detail in Section 2.5.

21. $y' = \epsilon y - \sigma y^3$, $\epsilon > 0$ and $\sigma > 0$. This equation occurs in the study of the stability of fluid flow.

22. $dy/dt = (\Gamma \cos t + T)y - y^3$, where Γ and T are constants. This equation also occurs in the study of the stability of fluid flow.

23. A differential equation of the form

$$\frac{dy}{dt} = A(t) + B(t)y + C(t)y^2 \qquad \text{(i)}$$

is called a **Riccati equation**. Such equations arise in optimal control theory.

(a) If y_1 is a known solution of (i), prove that the substitution $y = y_1 + v$ transforms (i) into a Bernoulli equation with $n = 2$.

(b) Solve the equation $\frac{dy}{dt} + 3ty = 4 - 4t^2 + y^2$, after showing that it has $y = 4t$ as a particular solution.

Mixed Practice. In each of Problems 24 through 36:

(a) List each of the following classes into which the equation falls: autonomous, separable, linear, exact, Bernoulli, homogeneous.

(b) Solve the equation. If it has more than one classification, solve it *two* different ways.

24. $(3x - y)\dfrac{dx}{dy} + (9y - 2x) = 0$

25. $1 = (3e^y - 2x)\dfrac{dy}{dx}$

26. $\dfrac{dy}{dx} - 4e^xy^2 = y$

27. $x\dfrac{dy}{dx} + (x + 1)y = x$

28. $\dfrac{dy}{dx} = \dfrac{xy^2 - \frac{1}{2}\sin 2x}{\left(1 - x^2\right)y}$

29. $\dfrac{\sqrt{x}}{y}\dfrac{dy}{dx} = 1$

30. $\left(5xy^2 + 5y\right) + \left(5x^2y + 5x\right)\dfrac{dy}{dx} = 0$

31. $2xy\dfrac{dy}{dx} + \ln x = -y^2 - 1$

32. $(2 - x)\dfrac{dy}{dx} = y + 2(2 - x)^5$

33. $x\dfrac{dy}{dx} = -\dfrac{1}{\ln x}$

34. $\dfrac{dx}{dy} = \dfrac{2xy + x^2}{3y^2 + 2xy}$

35. $4xy\dfrac{dy}{dx} = 8x^2 + 5y^2$

36. $\dfrac{dy}{dx} + y - \sqrt[4]{y} = 0$

CHAPTER SUMMARY

In this chapter we discuss a number of special solution methods for first order equations $dy/dt = f(t, y)$. The most important types of equations that can be solved analytically are **linear, separable,** and **exact** equations. Others, like Bernoulli and homogeneous equations, can be transformed into one of these. For equations that cannot be solved by symbolic analytic methods, it is necessary to resort to geometrical and numerical methods.

so that the substitution can be readily made. Doing so yields

$$\left(2(vy)^2y+4y^2(vy)+4y^3\right)=(vy)y^2\left[v+y\frac{dv}{dy}\right]. \tag{14}$$

Simplifying Eq. (14) yields

$$v^2+4v+4=vy\frac{dv}{dy}. \tag{15}$$

Separating the variables in Eq. (15) leads to

$$\frac{v}{(v+2)^2}dv=\frac{1}{y}dy. \tag{16}$$

We solve Eq. (16) by integrating both sides to arrive at the implicitly defined solution

$$\ln|v+2|+\frac{2}{v+2}=\ln|y|+c. \tag{17}$$

The solution of Eq. (12) is then obtained by resubstituting $v=\frac{x}{y}$ into Eq. (17):

$$\ln\left|\frac{x+2y}{y}\right|+\frac{2y}{x+2y}=\ln|y|+c, \qquad y\neq 0. \tag{18}$$

Remark. Looking back, had we used the substitution

$$y=ux, \qquad \frac{dy}{dx}=u+x\frac{du}{dx},$$

the algebra in Eq. (14) would have been slightly worse in that simplifying the left side would have entailed multiplying two binomials, whereas we only had to multiply a monomial times a binomial in Eq. (14) when using $x=vy$.

Bernoulli Differential Equations

A first order differential equation related to linear differential equations is the so-called *Bernoulli equation*, named after Jacob Bernoulli (1654–1705) and solved first by Leibnitz in 1696. Such an equation has the following form.

DEFINITION 2.7.2

Bernoulli Differential Equation. A differential equation of the form

$$\frac{dy}{dt}+q(t)y=r(t)y^n, \tag{19}$$

where n is any real number, is called a **Bernoulli equation**.

If $n=0$, then Eq. (19) is linear, and if $n=1$, then Eq. (19) is separable, linear, and homogeneous. For all other real values of n, Eq. (19) is not one of the forms studied thus far in the chapter.

To solve a Bernoulli equation when n is neither 0 nor 1, we shall make a substitution that reduces it to a linear equation that can subsequently be solved using the method of integrating factors. Specifically, we perform the following initial steps to transform Eq. (19) into a linear equation.

First divide Eq. (19) by y^n to obtain

$$y^{-n}\frac{dy}{dt} + q(t)y^{1-n} = r(t). \tag{20}$$

Define $u = y^{1-n}$, which is a function of t. Observe that

$$\frac{du}{dt} = (1-n)y^{-n}\frac{dy}{dt},$$

or equivalently,

$$y^{-n}\frac{dy}{dt} = \frac{1}{(1-n)}\frac{du}{dt}. \tag{21}$$

Substituting Eq. (21) into Eq. (20) yields

$$\frac{1}{1-n}\frac{du}{dt} + q(t)u(t) = r(t),$$

and subsequently,

$$\frac{du}{dt} + \underbrace{(1-n)q(t)}_{\text{Call this } p(t)}u(t) = \underbrace{(1-n)r(t)}_{\text{Call this } g(t)}, \tag{22}$$

which is a linear differential equation (in u).

Now solve Eq. (22) as you would any other linear differential equation. Once you obtain the solution $u(t)$, resubstitute $u(t) = y^{1-n}$ to determine the solution $y(t)$ of the original differential equation (19).

EXAMPLE 3

Solve the initial value problem

$$\frac{dy}{dt} + y = y^3, \quad y(0) = y_0, \tag{23}$$

where $-1 < y_0 < 1$. Determine the long-term behavior of the solution of Eq. (23) for such initial conditions.

To begin, divide both sides of the equation by y^3 to obtain

$$y^{-3}\frac{dy}{dt} + y^{-2} = 1.$$

Let $u = y^{-2}$ and observe that

$$\frac{du}{dt} = -2y^{-3}\frac{dy}{dt},$$

or equivalently,

$$-\frac{1}{2}\frac{du}{dt} = y^{-3}\frac{dy}{dt}. \tag{24}$$

Using the new variable u with Eq. (24) transforms the original equation into the linear equation

$$\frac{du}{dt} - 2u(t) = -2. \tag{25}$$

Solving Eq. (25) using the method of integrating factors leads to

$$u(t) = 1 + Ce^{2t}. \tag{26}$$