"Differential Equations: An Intro to Modern Methods + Applications" 3rd ed
J.R. Brannan + WE Boyce

10. $4y'' - 4y' + y = 16e^{t/2}$ (Compare with Problem 13 in Section 4.7.)

11. $2y'' + 3y' + y = t^2 + 3\sin t$

12. $y'' + y = 3\sin 2t + t\cos 2t$

13. $u'' + \omega_0^2 u = \cos \omega t, \quad \omega^2 \neq \omega_0^2$

14. $u'' + \omega_0^2 u = \cos \omega_0 t$

15. $y'' + y' + 4y = 2\sinh t$
Hint: $\sinh t = (e^t - e^{-t})/2$

16. $y'' - y' - 2y = \cosh 2t$
Hint: $\cosh t = (e^t + e^{-t})/2$

In each of Problems 17 through 22, find the solution of the given initial value problem.

17. $y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1$

18. $y'' + 4y = t^2 + 3e^t, \quad y(0) = 0, \quad y'(0) = 2$

19. $y'' - 2y' + y = te^t + 4, \quad y(0) = 1, \quad y'(0) = 1$

20. $y'' - 2y' - 3y = 3te^{2t}, \quad y(0) = 1, \quad y'(0) = 0$

21. $y'' + 4y = 3\sin 2t, \quad y(0) = 2, \quad y'(0) = -1$

22. $y'' + 2y' + 5y = 4e^{-t}\cos 2t, \quad y(0) = 1, \quad y'(0) = 0$

In each of Problems 23 through 30:

(a) Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

(b) Use a computer algebra system to find a particular solution of the given equation.

23. $y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin 3t$

24. $y'' + y = t(1 + \sin t)$

25. $y'' - 5y' + 6y = e^t\cos 2t + e^{2t}(3t + 4)\sin t$

26. $y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2\sin t$

27. $y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t\sin 2t$

28. $y'' + 4y = t^2\sin 2t + (6t + 7)\cos 2t$

29. $y'' + 3y' + 2y = e^t(t^2 + 1)\sin 2t + 3e^{-t}\cos t + 4e^t$

30. $y'' + 2y' + 5y = 3te^{-t}\cos 2t - 2te^{-2t}\cos t$

31. Consider the equation

$$y'' - 3y' - 4y = 2e^{-t} \qquad \text{(i)}$$

from Example 5. Recall that $y_1(t) = e^{-t}$ and $y_2(t) = e^{4t}$ are solutions of the corresponding homogeneous equation. Adapting the method of reduction of order (see the discussion preceding Problem 28 in Section 4.2), seek a solution of the nonhomogeneous equation of the form $Y(t) = v(t)y_1(t) = v(t)e^{-t}$, where $v(t)$ is to be determined.

(a) Substitute $Y(t)$, $Y'(t)$, and $Y''(t)$ into Eq. (i) and show that $v(t)$ must satisfy $v'' - 5v' = 2$.

(b) Let $w(t) = v'(t)$ and show that $w(t)$ must satisfy $w' - 5w = 2$. Solve this equation for $w(t)$.

(c) Integrate $w(t)$ to find $v(t)$ and then show that

$$Y(t) = -\tfrac{2}{5}te^{-t} + \tfrac{1}{5}c_1e^{4t} + c_2e^{-t}.$$

The first term on the right side is the desired particular solution of the nonhomogeneous equation. Note that it is a product of t and e^{-t}.

Nonhomogeneous Cauchy–Euler Equations. In each of Problems 32 through 35, find the general solution by using the change of variable $t = \ln x$ to transform the equation into one with constant coefficients (see the discussion preceding Problem 52 in Section 4.3).

32. $x^2y'' - 3xy' + 4y = \ln x$

33. $x^2y'' + 7xy' + 5y = x$

34. $x^2y'' - 2xy' + 2y = 3x^2 + 2\ln x$

35. $x^2y'' + xy' + 4y = \sin(\ln x)$

36. Determine the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin m\pi t,$$

where $\lambda > 0$ and $\lambda \neq m\pi$ for $m = 1, \ldots, N$.

37. In many physical problems, the nonhomogeneous term may be specified by different formulas in different time periods. As an example, determine the solution $y = \phi(t)$ of

$$y'' + y = \begin{cases} t, & 0 \le t \le \pi, \\ \pi e^{\pi - t}, & t > \pi, \end{cases}$$

satisfying the initial conditions $y(0) = 0$ and $y'(0) = 1$. Assume that y and y' are also continuous at $t = \pi$. Plot the nonhomogeneous term and the solution as functions of time.
Hint: First solve the initial value problem for $t \le \pi$; then solve for $t > \pi$, determining the constants in the latter solution from the continuity conditions at $t = \pi$.

38. Follow the instructions in Problem 37 to solve the differential equation

$$y'' + 2y' + 5y = \begin{cases} 1, & 0 \le t \le \pi/2, \\ 0, & t > \pi/2 \end{cases}$$

with the initial conditions $y(0) = 0$ and $y'(0) = 0$.

4.6 Forced Vibrations, Frequency Response, and Resonance

We will now investigate the situation in which a periodic external force is applied to a spring-mass system. The behavior of this simple system models that of many oscillatory systems with an external force due, for example, to a motor attached to the system. We will first consider the case in which damping is present and will look later at the idealized special case in which there is assumed to be no damping.

Forced Vibrations with Damping

Recall that the equation of motion for a damped spring-mass system with external forcing, $F(t)$, is

$$my'' + \gamma y' + ky = F(t), \tag{1}$$

where m, γ, and k are the mass, damping coefficient, and spring constant, respectively. Dividing through Eq. (1) by m puts it in the form

$$y'' + 2\delta y' + \omega_0^2 y = f(t), \tag{2}$$

where $\delta = \gamma/(2m)$, $\omega_0^2 = k/m$, and $f(t) = F(t)/m$. These definitions for δ and for ω_0 simplify important mathematical expressions that appear below as we analyze the behavior of solutions of Eq. (2).

The assumption that the external force is periodic means $f(t)$ involves a linear combination of $A\cos(\omega t)$ and $A\sin(\omega t)$ with frequency ω and amplitude A. While we could work with these forms individually, the ensuing analysis is, as we shall see, less complicated—and more informative—if we write the external force in the form of a complex-valued exponential: $f(t) = Ae^{i\omega t} = A\cos(\omega t) + iA\sin(\omega t)$, because it allows us to consider both trigonometric terms at once. Thus we wish to find the general solution of

$$y'' + 2\delta y' + \omega_0^2 y = Ae^{i\omega t}. \tag{3}$$

Note that the solutions $y_1(t)$ and $y_2(t)$ of the homogeneous equation corresponding to Eq. (3) depend on the roots λ_1 and λ_2 of the characteristic equation $\lambda^2 + 2\delta\lambda + \omega_0^2 = 0$. Note that m, γ, and k all positive imply that δ and ω_0^2 are also positive. Damped free vibrations are discussed in Section 4.4, and in Problem 51 in Section 4.3. Recall that λ_1 and λ_2 are either real and negative (when $\delta \geq \omega_0$) or are complex conjugates with a negative real part (when $0 < \delta < \omega_0$).

Because the exponent on the right-hand side of Eq. (3) is purely imaginary, its real part is zero. Consequently, the forcing function on the right-hand side of Eq. (3) cannot be a solution of the homogeneous equation. The correct form to assume for the particular solution using the method of undetermined coefficients is therefore $Y(t) = Ce^{i\omega t}$.

Substituting $Y(t)$ into Eq. (3) leads to

$$\left((i\omega)^2 + 2\delta(i\omega) + \omega_0^2\right) Ce^{i\omega t} = Ae^{i\omega t}.$$

Solving for the unknown coefficient in $Y(t)$ yields

$$C = \frac{A}{(i\omega)^2 + 2\delta(i\omega) + \omega_0^2},$$

so

$$Y(t) = \frac{1}{(i\omega)^2 + 2\delta(i\omega) + \omega_0^2} Ae^{i\omega t}. \tag{4}$$

The general solution of Eq. (3) is

$$y = y_c(t) + Y(t),$$

where $Y(t)$ is the particular solution in Eq. (4), and $y_c(t) = c_1 y_1(t) + c_2 y_2(t)$ is the general solution of the homogeneous equation with constants c_1 and c_2 depending on the initial conditions. Since the roots of $\lambda^2 + 2\delta\lambda + \omega_0^2 = 0$ are either real and negative or complex with negative real part, each of $y_1(t)$ and $y_2(t)$ contains an exponentially decaying term. As a consequence, $y_c(t) \to 0$ as $t \to 0$ and $y_c(t)$ is referred to as the **transient solution**. In many applications the transient solution is of little importance. Its primary purpose is to satisfy whatever initial conditions may be imposed. With increasing time, the energy put into the system by the initial displacement and velocity dissipates through the damping force. The motion then becomes the response of the system to the external force.

Note that without damping ($\delta = 0$), the effects of the initial conditions would persist for all time. This situation will be considered at the end of this section.

EXAMPLE 1

Consider the initial value problem

$$y'' + \tfrac{1}{8}y' + y = 3\cos(\omega t), \qquad y(0) = 2, \qquad y'(0) = 0. \tag{5}$$

Show plots of the solution for different values of the forcing frequency ω, and compare them with corresponding plots of the forcing function.

For this system we have $\delta = 1/16$ and $\omega_0 = 1$. The amplitude of the harmonic input, A, is equal to 3. The transient part of the solution of the forced problem (5) resembles the solution

$$y = e^{-t/16}\left(2\cos\frac{\sqrt{255}}{16}t + \frac{2}{\sqrt{255}}\sin\frac{\sqrt{255}}{16}t\right)$$

of the corresponding unforced problem that was discussed in Example 2 of Section 4.4. The graph of that solution was shown in Figure 4.4.7. Because the damping is relatively small, the transient solution of the problem (5) also decays fairly slowly.

Turning to the nonhomogeneous problem, since the external force is $3\cos(\omega t)$, we work with $f(t) = 3e^{i\omega t}$. From Eq. (4) we know

$$Y(t) = \frac{3}{(i\omega)^2 + i\omega/8 + 1}e^{i\omega t}. \tag{6}$$

Since $3\cos(\omega t) = \mathrm{Re}(3e^{i\omega t})$ the particular solution for Eq. (5) is the real part of Eq. (6). To identify the real part of Eq. (6) it helps if the denominator in Eq. (6) is real-valued. To bring this about, multiply both the numerator and denominator of Eq. (6) by the complex conjugate of the denominator.

$$\begin{aligned} Y(t) &= \frac{3e^{i\omega t}}{(i\omega)^2 + i\omega/8 + 1} \cdot \frac{(-i\omega)^2 - i\omega/8 + 1}{(-i\omega)^2 - i\omega/8 + 1} \\ &= \frac{3(\cos(\omega t) + i\sin(\omega t))\left(1 - \omega^2 - i\frac{\omega}{8}\right)}{(1-\omega^2)^2 + \frac{\omega^2}{64}}. \end{aligned}$$

Thus, the real part of Eq. (6) is

$$Y_{\mathrm{Re}}(t) = \mathrm{Re}Y(t) = \frac{3}{(1-\omega^2)^2 + \omega^2/64}\left((1-\omega^2)\cos(\omega t) + \frac{\omega}{8}\sin(\omega t)\right). \tag{7}$$

Figures 4.6.1, 4.6.2, and 4.6.3 show the solution of the forced problem (5) for $\omega = 0.3$, $\omega = 1$, and $\omega = 2$, respectively. The graph of the corresponding forcing function is shown (as a dashed curve) in each figure.

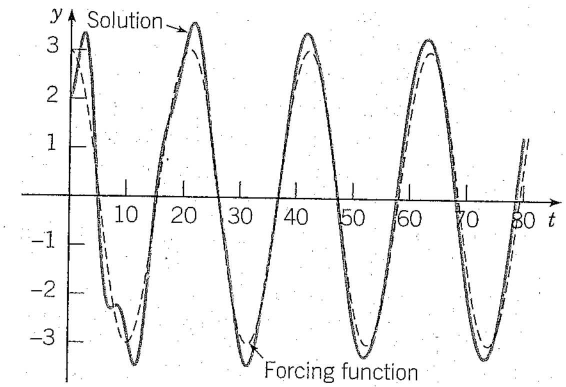

FIGURE 4.6.1 A forced vibration with damping; solution of $y'' + 0.125y' + y = 3\cos(3t/10)$, $y(0) = 2$, $y'(0) = 0$.

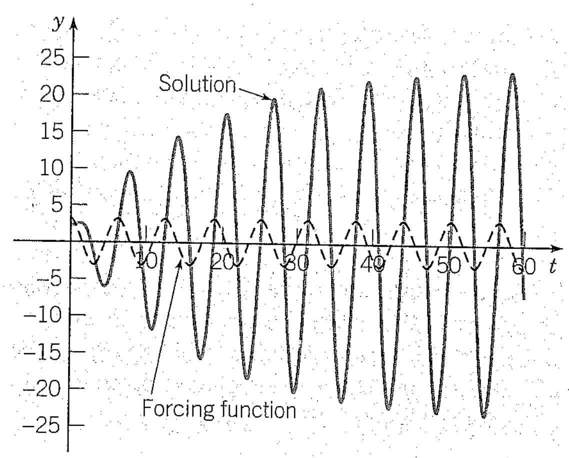

FIGURE 4.6.2 A forced vibration with damping; solution of $y'' + 0.125y' + y = 3\cos t$, $y(0) = 2$, $y'(0) = 0$.

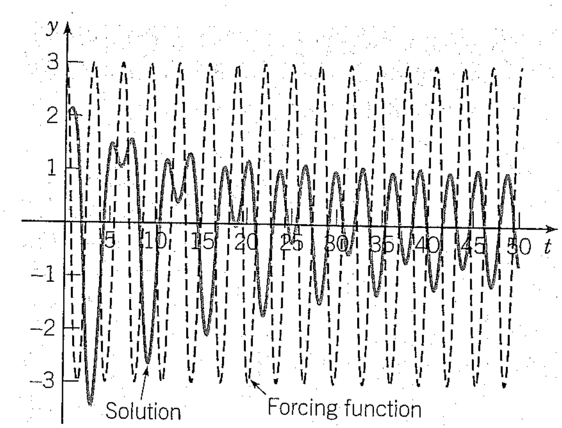

FIGURE 4.6.3 A forced vibration with damping; solution of $y'' + 0.125y' + y = 3\cos 2t$, $y(0) = 2$, $y'(0) = 0$.

The solutions in Figures 4.6.1, 4.6.2, and 4.6.3 show three different behaviors. In each case the solution does not die out as t increases but persists indefinitely, or at least as long as the external force is applied. From Eq. (7) we see that each solution represents a steady oscillation with the same frequency as the external force. For these reasons the particular solution to a damped, harmonically forced system is called the **steady-state solution**, the **steady-state response**, the **steady-state output**, or the **forced response**.

In general, in the real-valued case with $f(t) = A\cos(\omega t)$, the real-valued particular solution is the real part of Eq. (4). Now, apply to Eq. (4) the same steps that were just applied to Eq. (6); in this way we obtain the following more general expression for the real part of Eq. (4):

$$Y_{\mathrm{Re}}(t) = \mathrm{Re}\,Y(t) = A\frac{(\omega_0^2 - \omega^2)\cos(\omega t) + 2\delta\omega\sin(\omega t)}{(\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2}. \tag{8}$$

The Frequency Response Function

In Example 1, even though the forcing function is a pure cosine, $\cos(\omega t)$, the forced response involves both $\sin(\omega t)$ and $\cos(\omega t)$. However, when considering the external force as a complex-valued exponential, Eq. (4) tells us that the forced output is directly proportional to the forced input:

$$\frac{Y(t)}{Ae^{i\omega t}} = \frac{1}{(i\omega)^2 + 2\delta(i\omega) + \omega_0^2}.$$

This quotient is referred to as the **frequency response** of the system. As the frequency response depends on the frequency ω (and δ and ω_0)—but not on t—it is commonly defined as

$$G(i\omega) = \frac{1}{(i\omega)^2 + 2\delta(i\omega) + \omega_0^2} = \frac{1}{(i\omega + \delta)^2 + \omega_0^2 - \delta^2}. \tag{9}$$

To continue the analysis of the frequency response function, it is convenient to represent the $G(i\omega)$ in Eq. (9) in its complex exponential form (see Figure 4.6.4),

$$G(i\omega) = |G(i\omega)|\,e^{-i\phi(\omega)} = |G(i\omega)|\left(\cos(\phi(\omega)) - i\sin(\phi(\omega))\right), \tag{10}$$

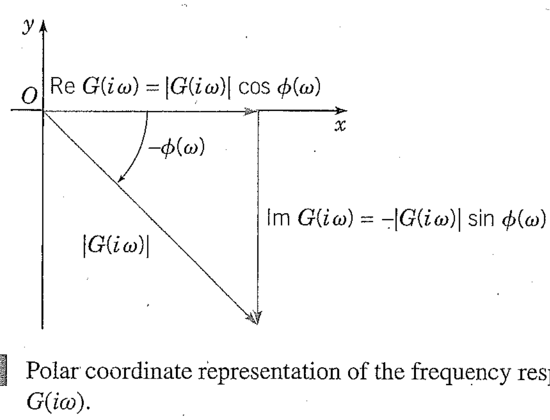

FIGURE 4.6.4 Polar coordinate representation of the frequency response function $G(i\omega)$.

where the **gain** of the frequency response is

$$|G(i\omega)| = \left(G(i\omega)\overline{G(i\omega)}\right)^{1/2} = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2}} \tag{11}$$

and the **phase** of the frequency response is the angle

$$\phi(\omega) = \arccos\left(\frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\delta^2\omega^2}}\right). \tag{12}$$

Using Eq. (10), the particular solution (4) is

$$Y(t) = G(i\omega)Ae^{i\omega t} = |G(i\omega)|\, e^{-i\phi(\omega)}Ae^{i\omega t} = A\,|G(i\omega)|\, e^{i(\omega t - \phi(\omega))}. \tag{13}$$

EXAMPLE 2

Find the gain and phase for each of the three response functions found in Example 1.

Recall that $\omega_0 = 1$ and $\delta = 1/16$. The three cases are $\omega = 0.3$, $\omega = 1$, and $\omega = 2$.

Classification of the external force as low- or high-frequency is done relative to the forcing frequency ω_0. For example, the case with $\omega/\omega_0 = 0.3$ is a low-frequency force. The steady-state response is

$$\begin{aligned} Y(t) &= 3\frac{0.91\cos(0.3t) + \frac{0.3}{8}\sin(0.3t)}{0.91^2 + 0.09/64} \\ &\approx 3.29111\cos(0.3t) + 0.13562\sin(0.3t) \approx 3.2923\cos(0.3t - 0.04119). \end{aligned}$$

That the gain is a little larger than the amplitude of the input and the phase is small are consistent with the graph shown in Figure 4.6.1.

For the comparatively high-frequency case, $\omega/\omega_0 = 2$, the particular solution is

$$\begin{aligned} Y(t) &= 3\frac{-3\cos(2t) + \frac{1}{4}\sin(2t)}{9 + \frac{1}{16}} \\ &\approx 0.99310\cos(2t) + 0.08276\sin(2t) \approx 0.99655\cos(2t - 3.0585). \end{aligned}$$

In this case the amplitude of the steady forced response is approximately one-third the amplitude of the harmonic input and the phase between the excitation and the response is approximately π. These findings are consistent with the particular solution plotted in Figure 4.6.3.

In the third case, using Eq. (8) with $\omega/\omega_0 = 1$, the steady-state response is

$$\begin{aligned} Y(t) &= \frac{3}{1/64}\left((1-1)\cos(t) + \frac{1}{8}\sin(t)\right) \\ &= 24\sin(t) = 24\cos(t - \pi/2). \end{aligned}$$

Here, the gain is much larger—8 times the amplitude of the harmonic input—and the phase is exactly $\pi/2$ relative to the external force.

The explicit formulas for the gain factor and phase shift given in Eqs. (11) and (12) are rather complicated. The three cases considered in Examples 1 and 2 illustrate the

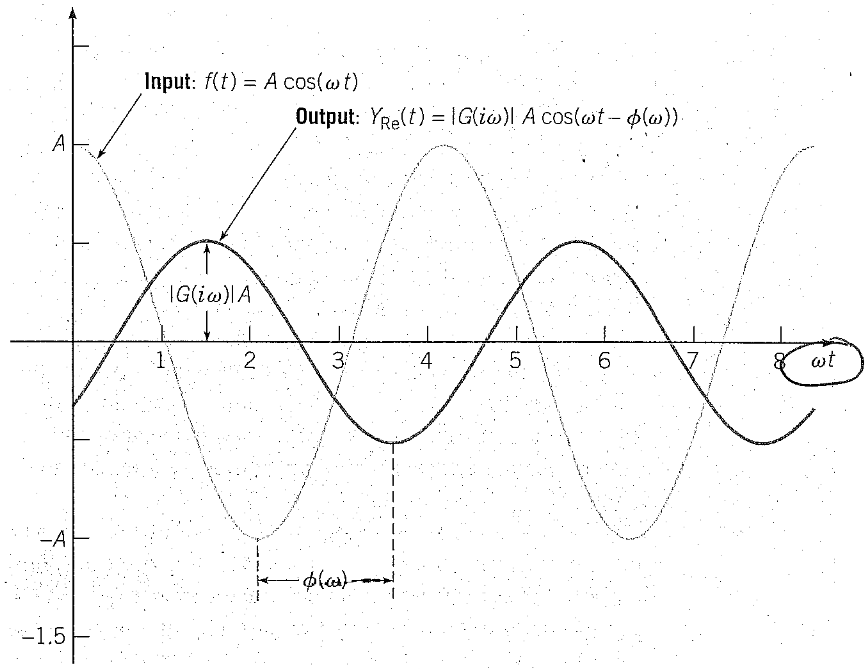

FIGURE 4.6.5 The steady-state response $Y_{\text{Re}} = |G(i\omega)|A\cos(\omega t - \phi(\omega))$ of a spring-mass system due to the harmonic input $f(t) = A\cos\omega t$.

two ways the harmonic input is modified as it passes through a spring-mass system (see Figure 4.6.5):

1. The amplitude of the output equals the amplitude of the harmonic input amplified or attenuated by the gain factor, $|G(i\omega)|$.
2. There is a phase shift in the steady-state output of magnitude $\phi(\omega)$ relative to the harmonic input.

Our next objective is to understand better how the gain $|G(i\omega)|$ and the phase shift $\phi(\omega)$ depend on the frequency of the harmonic input. For low-frequency excitation, that is, as $\omega \to 0^+$, it follows from Eq. (11) that $|G(i\omega)| \to 1/\omega_0^2 = m/k$. At the other extreme, for very high-frequency excitation, Eq. (11) implies that $|G(i\omega)| \to 0$ as $\omega \to \infty$.

The case with $\omega/\omega_0 = 1$ in Example 2 suggests that the gain can have a maximum at an intermediate value of ω. To find this maximum point, find where the derivative of $|G(i\omega)|$ with respect to ω is zero. You will find that the maximum amplitude occurs when $\omega = \omega_{\text{max}}$, where

$$\omega_{\text{max}}^2 = \omega_0^2 - 2\delta^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2\left(1 - \frac{\gamma^2}{2mk}\right). \tag{14}$$

Note that $0 < \omega_{\text{max}} < \omega_0$ and, when the damping coefficient, γ, is small, ω_{max} is close to ω_0. The maximum value of the gain is

$$|G(i\omega_{\text{max}})| = \frac{m}{\gamma\omega_0\sqrt{1 - (\gamma^2/4mk)}} \approx \frac{m}{\gamma\omega_0}\left(1 + \frac{\gamma^2}{8mk}\right), \tag{15}$$

where the last expression is an approximation for small γ.

If $\gamma^2/mk > 2$, then ω_{max}, as given by Eq. (14), is imaginary. In this case, which is identified as highly damped, the maximum value of the gain occurs for $\omega = 0$, and $|G(i\omega)|$ is a monotone decreasing function of ω. Recall that critical damping occurs when $\gamma^2/mk = 4$.

For small values of γ, if follows from Eq. (15) that $|G(i\omega_{\text{max}})| \approx m/\gamma\omega_0$. Thus, for lightly damped systems, the gain $|G(i\omega)|$ is large when $\omega/\omega_0 \approx 1$. Moreover the smaller the value of γ, the more pronounced is this effect.

Resonance is the physical tendency of solutions to periodically forced systems to have a steady-state response that oscillates with a much greater amplitude than the input. The specific frequency at which the amplitude of the steady state response has a local maximum is called the **resonant frequency** of the system.

Resonance can be an important design consideration; it can be good or bad, depending on the circumstances. It must be taken seriously in the design of structures, such as buildings and bridges, where it can produce instabilities that might lead to the catastrophic failure of the structure. On the other hand, resonance can be put to good use in the design of instruments, such as seismographs, that are intended to detect weak periodic incoming signals.

The phase angle ϕ also depends in an interesting way on ω. For ω near zero, it follows from Eq. (12) that $\cos(\phi) \approx 1$. Thus $\phi \approx 0$, and the response is nearly in phase with the excitation. That is, they rise and fall together and, in particular, they assume their respective maxima nearly together and their respective minima nearly together.

For the resonant frequency, $\omega = \omega_0$, we find that $\cos(\phi) = 0$, so $\phi = \pi/2$. In this case the response lags behind the excitation by $\pi/2$, that is, the peaks of the response occur $\pi/2$ later than the peaks of the excitation, and similarly for the valleys.

Finally, for ω very large (relative to ω_0), we have $\cos(\phi) \approx -1$. Here, $\phi \approx \pi$, so the response is nearly out of phase with the excitation. In these cases the response is minimum when the excitation is maximum, and vice versa.

We conclude this discussion of frequency response, gain, phase, and resonance by looking at typical graphs of the gain and phase. Figures 4.6.6 and 4.6.7 plot the normalized

Ask yourself: why are the axes rescaled in these ways?

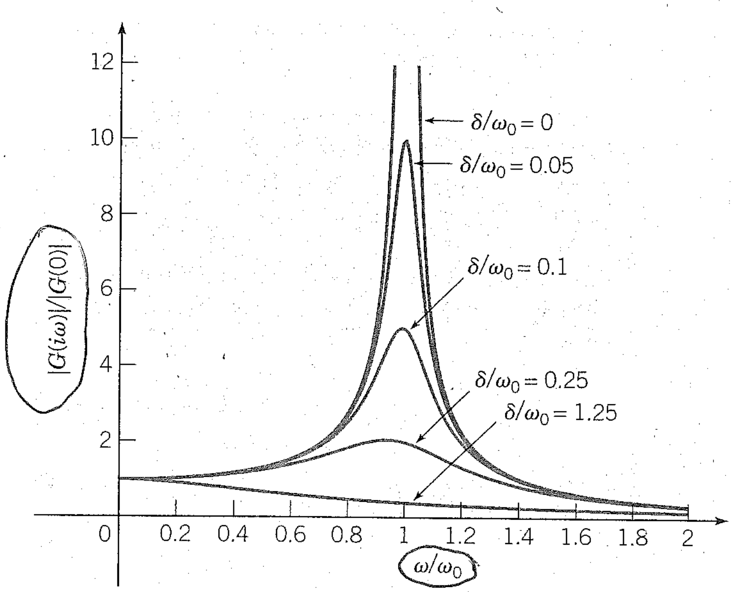

FIGURE 4.6.6 Gain function $|G(i\omega)|$ for the damped spring-mass system: $\delta/\omega_0 = \gamma/2\sqrt{mk}$.

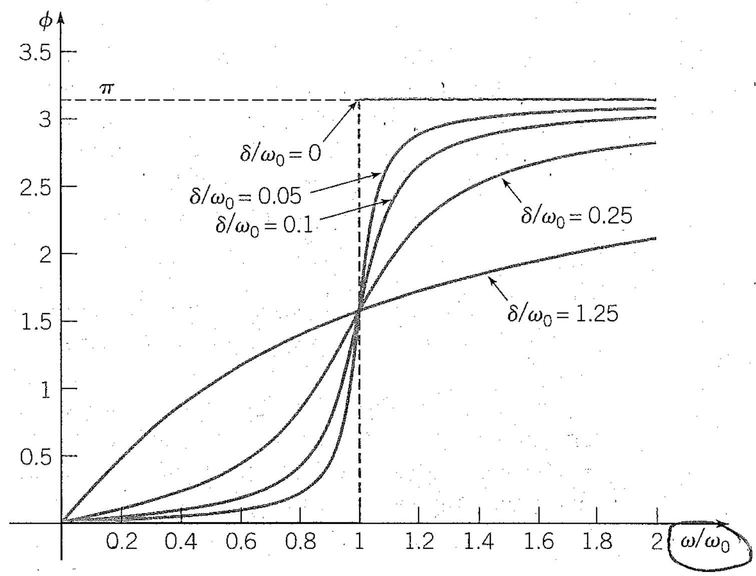

FIGURE 4.6.7 Phase function $\phi(\omega)$ for the damped spring-mass system: $\delta/\omega_0 = \gamma/2\sqrt{mk}$.

gain, $|G(i\omega)|/G(0)$, and the phase, $\phi(\omega)$, versus the normalized wavelength, ω/ω_0. With these normalizations, each frequency response curve in Figure 4.6.6 starts at height 1 when $\omega/\omega_0 = 0$. For heavily damped systems ($\gamma^2/4m > 4$), the response decreases for all $\omega > 0$. As the damping decreases, the frequency response acquires a maximum at $\omega/\omega_0 = 1$. The size of the gain increases as $\delta \to 0^+$.

In a similar way, the phase is always 0 when $\omega = 0$, $\pi/2$ when $\omega/\omega_0 = 1$, and approaches π as $\omega/\omega_0 \to \infty$, as shown in Figure 4.6.7. Notice how the transition from $\phi \approx 0$ to $\phi \approx \pi$ becomes more rapid as the damping decreases.

To conclude this introduction to the frerquency response function for damped systems, we point out how Figure 4.6.6 illustrates the usefulness of dimensionless variables. It is easy to verify that each of the quantities $|G(i\omega)|/|G(0)| = \omega^2|G(i\omega)|$, ω/ω_0, and $\delta/\omega_0 = \gamma/(2\sqrt{mk})$ is dimensionless. The importance of this observation can be seen in that the number of parameters in the problems has been reduced from the five that appear in Eq. (3)—m, γ, k, A, and ω—to the three that are in Eq. (3), namely, δ, ω_0, and ω. Thus this one family of curves, of which a few are shown in Figure 4.6.6, describes the response-versus-frequency behavior of the gain factor for all systems governed by Eq. (3). Likewise, Figure 4.6.7 shows representative curves describing the response-versus-frequency behavior of the phase shift for any solution to Eq. (3).

Forced Vibrations Without Damping

Notice that while Figures 4.6.6 and 4.6.7 include curves labeled as $\delta/\omega_0 = 0$, these curves are not governed by the formulas for $G(i\omega)$ and $\phi(\omega)$ given in this section. We conclude with a discussion of the limiting case when there is no damping.

We now assume $\gamma = 0$ in Eq. (1) so that $\delta = \gamma/2m = 0$ in Eq. (2), thereby obtaining the equation of motion of an undamped forced oscillator

$$y'' + \omega_0^2 y = A\cos\omega t, \qquad (16)$$

where we have assumed that $f(t) = A\cos\omega t$. The form of the general solution of Eq. (16) is different, depending on whether the forcing frequency ω is different from or equal to the natural frequency $\omega_0 = \sqrt{k/m}$ of the unforced system. First consider the case $\omega \neq \omega_0$; then the general solution of Eq. (16) is

$$y = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + \frac{A}{\left(\omega_0^2 - \omega^2\right)} \cos\omega t. \tag{17}$$

The constants c_1 and c_2 are determined by the initial conditions. The resulting motion is, in general, the sum of two periodic motions of different frequencies (ω_0 and ω) and amplitudes.

It is particularly interesting to suppose that the mass is initially at rest, so the initial conditions are $y(0) = 0$ and $y'(0) = 0$. Then the energy driving the system comes entirely from the external force, with no contribution from the initial conditions. In this case, it turns out that the constants c_1 and c_2 in Eq. (17) are given by

$$c_1 = -\frac{A}{\left(\omega_0^2 - \omega^2\right)}, \qquad c_2 = 0, \tag{18}$$

and the solution of Eq. (16) is

$$y = \frac{A}{\left(\omega_0^2 - \omega^2\right)}(\cos\omega t - \cos\omega_0 t). \tag{19}$$

This is the sum of two periodic functions of different periods but the same amplitude. Making use of the trigonometric identities for $\cos(A \pm B)$ with $A = (\omega_0 + \omega)t/2$ and $B = (\omega_0 - \omega)t/2$, we can write Eq. (19) in the form

$$y = \left[\frac{2A}{\left(\omega_0^2 - \omega^2\right)} \sin\frac{(\omega_0 - \omega)t}{2}\right] \sin\frac{(\omega_0 + \omega)t}{2}. \tag{20}$$

If $|\omega_0 - \omega|$ is small, then $\omega_0 + \omega$ is much greater than $|\omega_0 - \omega|$. Hence $\sin((\omega_0 + \omega)t/2)$ is a rapidly oscillating function compared to $\sin((\omega_0 - \omega)t/2)$. Thus the motion is a rapid oscillation with frequency $(\omega_0 + \omega)/2$ but with a slowly varying sinusoidal amplitude

$$\frac{2A}{\left|\omega_0^2 - \omega^2\right|} \left|\sin\frac{(\omega_0 - \omega)t}{2}\right|.$$

This type of motion, possessing a periodic variation of amplitude, exhibits what is called a **beat**. For example, such a phenomenon occurs in acoustics when two tuning forks of nearly equal frequency are excited simultaneously. In this case, the periodic variation of amplitude is quite apparent to the unaided ear. In electronics, the variation of the amplitude with time is called **amplitude modulation**.

EXAMPLE 3

Solve the initial value problem

$$y'' + y = 0.5\cos 0.8t, \qquad y(0) = 0, \quad y'(0) = 0, \tag{21}$$

and plot the solution.

In this case, $\omega_0 = 1$, $\omega = 0.8$, and $A = 0.5$, so from Eq. (20) the solution of the given problem is

$$y = 2.77778 \sin 0.1t \sin 0.9t. \tag{22}$$

A graph of this solution is shown in Figure 4.6.8.

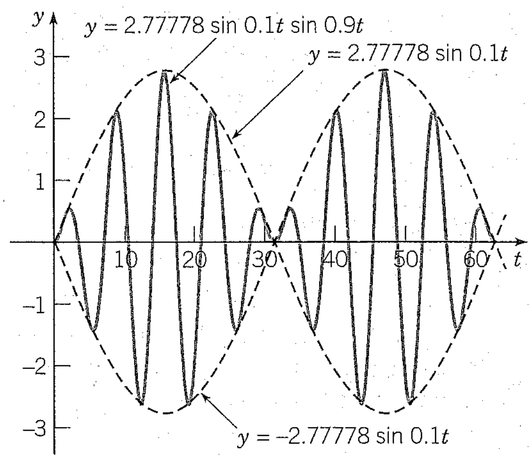

FIGURE 4.6.8 A beat; solution of $y'' + y = 0.5 \cos 0.8t$, $y(0) = 0$, $y'(0) = 0$; $y = 2.77778 \sin 0.1t \sin 0.9t$.

The amplitude variation has a slow frequency of 0.1 and a corresponding slow period of 20π. Note that a half-period of 10π corresponds to a single cycle of increasing and then decreasing amplitude. The displacement of the spring-mass system oscillates with a relatively fast frequency of 0.9, which is only slightly less than the natural frequency ω_0.

Now imagine that the forcing frequency ω is further increased, say, to $\omega = 0.9$. Then the slow frequency is halved to 0.05, and the corresponding slow half-period is doubled to 20π. The multiplier 2.7778 also increases substantially, to 5.2632. However the fast frequency is only marginally increased, to 0.95. Can you visualize what happens as ω takes on values closer and closer to the natural frequency $\omega_0 = 1$?

Now let us return to Eq. (16) and consider the case of resonance, where $\omega = \omega_0$, that is, the frequency of the forcing function is the same as the natural frequency of the system. Then the nonhomogeneous term $A \cos \omega t$ is a solution of the homogeneous equation. In this case, the solution of Eq. (16) is

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{A}{2\omega_0} t \sin \omega_0 t. \tag{23}$$

Because of the term $t \sin \omega_0 t$, the solution (23) predicts that the motion will become unbounded as $t \to \infty$ regardless of the values of c_1 and c_2; see Figure 4.6.9 for a typical example. Of course, in reality, unbounded oscillations do not occur. As soon as y becomes large, the mathematical model on which Eq. (16) is based is no longer valid, since the assumption that the spring force depends linearly on the displacement requires that y be

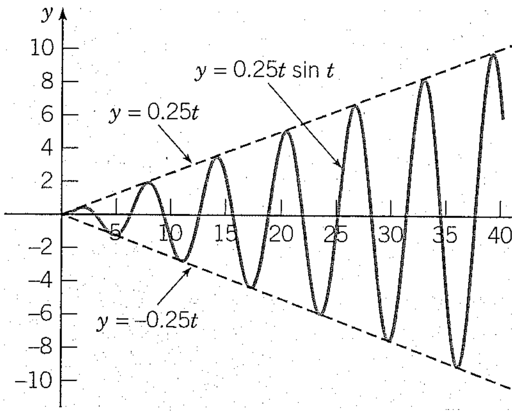

FIGURE 4.6.9 Resonance; solution of $y'' + y = 0.5 \cos t$, $y(0) = 0$, $y'(0) = 0$; $y = 0.25t \sin t$.

small. As we have seen, if damping is included in the model, the predicted motion remains bounded. However the response to the input function $A \cos \omega t$ may be quite large if the damping is small and ω is close to ω_0.

PROBLEMS

In each of Problems 1 through 4, write the given expression as a product of two trigonometric functions of different frequencies.

1. $\cos 11t - \cos 3t$
2. $\sin 7t - \sin 4t$
3. $\cos 7\pi t + \cos 2\pi t$
4. $\sin 9t + \sin 4t$

5. A mass weighing 4 pounds (lb) stretches a spring 1.5 in. The mass is displaced 12 in. in the positive direction from its equilibrium position and released with no initial velocity. Assuming that there is no damping and that the mass is acted on by an external force of $7 \cos 3t$ lb, formulate the initial value problem describing the motion of the mass.

6. A mass of 4 kg stretches a spring 8 cm. The mass is acted on by an external force of $8 \sin(t/2)$ newtons (N) and moves in a medium that imparts a viscous force of 4 N when the speed of the mass is 2 cm/s. If the mass is set in motion from its equilibrium position with an initial velocity of 16 cm/s, formulate the initial value problem describing the motion of the mass.

7. **(a)** Find the solution of Problem 5.
(b) Plot the graph of the solution.
(c) If the given external force is replaced by a force $A \exp(i\omega t)$ of frequency ω, find the frequency response $G(i\omega)$, the gain $|G(i\omega)|$, and the phase $\phi(\omega) = -\arg(G(i\omega))$. Then find the value of ω for which resonance occurs.

8. **(a)** Find the solution of the initial value problem in Problem 6.
(b) Identify the transient and steady-state parts of the solution.
(c) Plot the graph of the steady-state solution.
(d) If the given external force is replaced by a force $A \exp(i\omega t)$ of frequency ω, find the frequency response $G(i\omega)$, the gain $|G(i\omega)|$, and the phase $\phi(\omega) = -\arg(G(i\omega))$. Then find the value of ω for which the gain is maximum. Plot the graphs of $|G(i\omega)|$ and $\phi(\omega)$.

9. If an undamped spring-mass system with a mass that weighs 12 lb and a spring constant 2 lb/in. is suddenly set in motion at $t = 0$ by an external force of $15 \cos 7t$ lb, determine the position of the mass at any time and draw a graph of the displacement versus t.

10. A mass that weighs 8 lb stretches a spring 24 in. The system is acted on by an external force of $4 \sin 4t$ lb. If the mass is pulled down 6 in. and then released, determine the position of the mass at any time. Determine the first four times at which the velocity of the mass is zero.

11. A spring is stretched 6 in. by a mass that weighs 8 lb. The mass is attached to a dashpot mechanism that has a damping constant of 0.25 lb·s/ft and is acted on by an external force of $3 \cos 2t$ lb.
(a) Determine the steady-state response of this system.
(b) If the given mass is replaced by a mass m, determine the value of m for which the amplitude of the steady-state response is maximum.

12. A spring-mass system has a spring constant of 3 N/m. A mass of 2 kg is attached to the spring, and the

motion takes place in a viscous fluid that offers a resistance numerically equal to the magnitude of the instantaneous velocity. If the system is driven by an external force of $(12\cos 3t - 8\sin 3t)$ N, determine the steady-state response.

13. Furnish the details in determining when the gain function given by Eq. (10) is maximum, that is, show that ω^2_{max} and $|G(i\omega_{max})|$ are given by Eqs. (14) and (15), respectively.

14. Find the solution of the initial value problem

$$y'' + y = F(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where

$$F(t) = \begin{cases} At, & 0 \le t \le \pi, \\ A(2\pi - t), & \pi < t \le 2\pi, \\ 0, & 2\pi < t. \end{cases}$$

Hint: Treat each time interval separately, and match the solutions in the different intervals by requiring that y and y' be continuous functions of t.

15. A series circuit has a capacitor of 0.25 microfarad, a resistor of 5×10^3 ohms, and an inductor of 1 henry. The initial charge on the capacitor is zero. If a 9-volt battery is connected to the circuit and the circuit is closed at $t = 0$, determine the charge on the capacitor at $t = 0.001$ s, at $t = 0.01$ s, and at any time t. Also determine the limiting charge as $t \to \infty$.

16. Consider a vibrating system described by the initial value problem

$$y'' + 0.25y' + 2y = 2\cos\omega t,$$
$$y(0) = 0, \quad y'(0) = 2.$$

(a) Determine the steady-state part of the solution of this problem.
(b) Find the gain function $|G(i\omega)|$ of the system.
(c) Plot $|G(i\omega)|$ and $\phi(\omega) = -\arg(G(i\omega))$ versus ω.
(d) Find the maximum value of $|G(i\omega)|$ and the frequency ω for which it occurs.

17. Consider the forced but undamped system described by the initial value problem

$$y'' + y = 3\cos\omega t, \qquad y(0) = 0, \quad y'(0) = 0.$$

(a) Find the solution $y(t)$ for $\omega \neq 1$.
(b) Plot the solution $y(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Describe how the response $y(t)$ changes as ω varies in this interval. What happens as ω takes on values closer and closer to 1? Note that the natural frequency of the unforced system is $\omega_0 = 1$.

18. Consider the vibrating system described by the initial value problem

$$y'' + y = 3\cos\omega t, \qquad y(0) = 1, \quad y'(0) = 1.$$

(a) Find the solution for $\omega \neq 1$.
(b) Plot the solution $y(t)$ versus t for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$. Compare the results with those of Problem 17, that is, describe the effect of the nonzero initial conditions.

19. For the initial value problem in Problem 18, plot y' versus y for $\omega = 0.7$, $\omega = 0.8$, and $\omega = 0.9$, that is, draw the phase plot of the solution for these values of ω. Use a t interval that is long enough, so the phase plot appears as a closed curve. Mark your curve with arrows to show the direction in which it is traversed as t increases.

Problems 20 through 22 deal with the initial value problem

$$y'' + 0.125y' + 4y = f(t), \qquad y(0) = 2, \quad y'(0) = 0.$$

In each of these problems:
(a) Plot the given forcing function $f(t)$ versus t, and also plot the solution $y(t)$ versus t on the same set of axes. Use a t interval that is long enough, so the initial transients are substantially eliminated. Observe the relation between the amplitude and phase of the forcing term and the amplitude and phase of the response. Note that $\omega_0 = \sqrt{k/m} = 2$.
(b) Draw the phase plot of the solution, that is, plot y' versus y.

20. $f(t) = 3\cos(t/4)$

21. $f(t) = 3\cos 2t$

22. $f(t) = 3\cos 6t$

23. A spring-mass system with a hardening spring (Section 4.1) is acted on by a periodic external force. In the absence of damping, suppose that the displacement of the mass satisfies the initial value problem

$$y'' + y + 0.2y^3 = \cos\omega t, y(0) = 0, \qquad y'(0) = 0.$$

(a) Let $\omega = 1$ and plot a computer-generated solution of the given problem. Does the system exhibit a beat?
(b) Plot the solution for several values of ω between $\frac{1}{2}$ and 2. Describe how the solution changes as ω increases.

24. Suppose that the system of Problem 23 is modified to include a damping term and that the resulting initial value problem is

$$y'' + 0.2y' + y + 0.2y^3 = \cos\omega t, \quad y(0) = 0, \quad y'(0) = 0.$$

(a) Plot a computer-generated solution of the given problem for several values of ω between $\frac{1}{2}$ and 2, and estimate the amplitude, say, $G_H(\omega)$, of the steady response in each case.
(b) Using the data from part (a), plot the graph of $G_H(\omega)$ versus ω. For what frequency ω is the amplitude greatest?
(c) Compare the results of parts (a) and (b) with the corresponding results for the linear spring.