Some Study Notes for the Second Midterm

Here are some things that you need to think about. If you don't know how to do them, make sure that you can before the midterm comes around — ask your TA, ask me, etc. For the "solve the following ODE" problems, you can always use wolframalpha to tell you what the right answer is, if you get stuck. (Obviously, if you think you have an answer then you simply check to see if it solves the ODE and the initial conditions. . .)

1. **This will be in HW # 3 somehow; you're not going to be tested on this.** Remember the Picard-Lindelöf Theorem? The existence and uniqueness theorem for $y' = f(x, y)$ where we assumed f is continuous in x and Lipschitz in y? (The one whose proof involved Picard iterates/the Picard map.) The proof I presented was for a single ODE. Now that we're considering systems of ODEs, we need an existence and uniqueness theorem for $\vec{X} = \vec{F}(t, \vec{X})$. Make sure that you understand how the proof for one ODE is modified to be a proof for a system of ODEs. If you get stuck, see pages 387-394 in Hirsch, Smale, & Devaney — they do it for $\vec{X} = \vec{F}(\vec{X})$. Figure out how to modify their proof for autonomous systems (no t-dependence in the vector field) to make a proof for nonautonomous systems (the vector field can depend on t).

2. **This will be in HW # 3 somehow; you're not going to be tested on this.** The existence & uniqueness theorem assumed that $f(x, y)$ is Lipschitz in y. Convince yourself that this is more than you really need — that being locally Lipschitz in y would be good enough. And that being C^1 in y would be good enough. (I.e. $\frac{\partial f}{\partial y}$ exists and is continuous.)

3. **This will be in HW # 3 somehow; you're not going to be tested on this.** Problems 1, 2, and 3 from the second problem set referred to a solution matrix $\mathcal{X}(t)$. What determines the interval (a, b) for which this matrix is defined? Could you have that the first column is a solution on an interval (a_1, b_1) and the second column is a solution on (a_2, b_2) where these intervals are not the same and so $(a, b) = (a_1, b_1) \cap (a_2, b_2)$? How does your answer depend on whether it's the solution matrix from $\vec{X}' = A\vec{X}$, $\vec{X}' = A(t)\,\vec{X}$, or from $x'' + a_1(t)\,x' + a_0(t)x = 0$?

4. **This will be in HW # 3 somehow; you're not going to be tested on this.** Now that you have an existence & uniqueness theorem for systems, you should know how to prove from the theorem that $\vec{X}' = A\vec{X}$ and $x^{(n)} + a_{n-1}\,x^{(n-1)} + \ldots a_1\,x' + a_0 x = 0$ have solutions on $\mathbb{R}$. And given an ODE of the form $x'' + a_1(t)\,x' + a_0(t)\,x = g(t)$ you should be able to predict the interval of existence of solutions based on properties of the coefficients $a_1(t)$ and $a_0(t)$ and the forcing $g(t)$.

5. When I graphed solutions of $y' = f(x, y)$, if two graphs crossed then this meant that there were two different solutions that coincided at some point (x_0, y_0). This violates uniqueness of solutions and such crossing cannot happen if f is continuous in x and Lipschitz in y. When doing phase plane plots of $\vec{X}' = A\vec{X} \in \mathbb{R}^2$, I drew them so that the curves never crossed. In general, when doing a phase plane plot of a solution of $\vec{X} = \vec{F}(\vec{X}) \in \mathbb{R}^2$ what I'm doing is projecting a trajectory $(\vec{X}(t), t)$ that lives in $\mathbb{R}^2 \times \mathbb{R}$

onto $\mathbb{R}^2$: the time dynamics are indicated by putting arrows onto the projection of the curve. You need to understand why it is that the curves in the phase plane plot for $\vec{X}' = A\vec{X}$ can't cross — how it would violate the existence & uniqueness theorem. And, of course, if it's a general system $\vec{X} = \vec{F}(\vec{X}) \in \mathbb{R}^2$ then the curves in the phase plane plot can't cross if $\vec{F}$ is "nice". (How nice is nice?)

6. Convince yourself that the phase plane plot for $\vec{X} = \vec{F}(t, \vec{X}) \in \mathbb{R}^2$ can include curves that cross without contradicting the existence & uniqueness theorem.

7. Find the general solution to each of the following ODEs

$$
\begin{aligned}
x'' + 25x &= 0 \\
x'' - 25x &= 0 \\
x'' - 2x' + x &= 0 \\
x'' + 2x' + x &= 0 \\
x'' + x' + x &= 0
\end{aligned}
$$

8. For each of the above, find the solution that satisfies $x(0) = 1$ and $x'(0) = 0$. Find the solution that satisfies $x(0) = 0$ and $x'(0) = 1$.

9. For $\epsilon \neq 0$ solve the initial value problem

$$x_\epsilon'' - 2\,x_\epsilon' + (1 - \epsilon^2)\,x_\epsilon = 0, \qquad x_\epsilon(0) = 0, \quad x_\epsilon'(0) = 1.$$

Compute the limit $x(t) := \lim_{\epsilon \to 0} x_\epsilon(t)$ and show that the limit is a solution of

$$x'' - 2\,x' + x = 0, \qquad x(0) = 0, \quad x'(0) = 1.$$

10. Find the general solution to each of the following ODEs

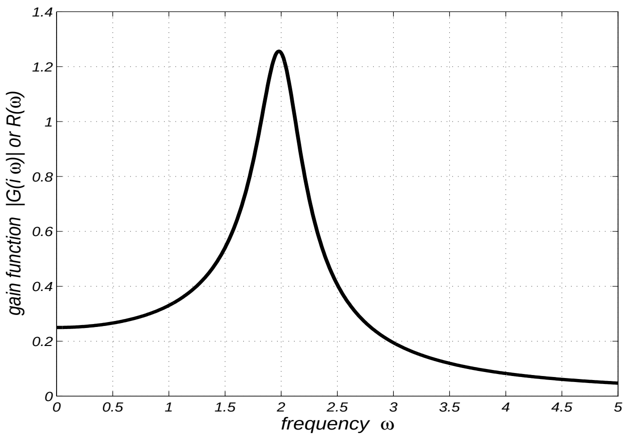
$$
\begin{aligned}
x'' + 25x &= e^{5t} + \sin(t) + \cos(5t) \\
x'' - 25x &= e^{5t} + te^{-5t} + \sin(5t) \\
x'' - 2x' + x &= t + e^t \\
x'' + 2x' + x &= 9 - t^3
\end{aligned}
$$

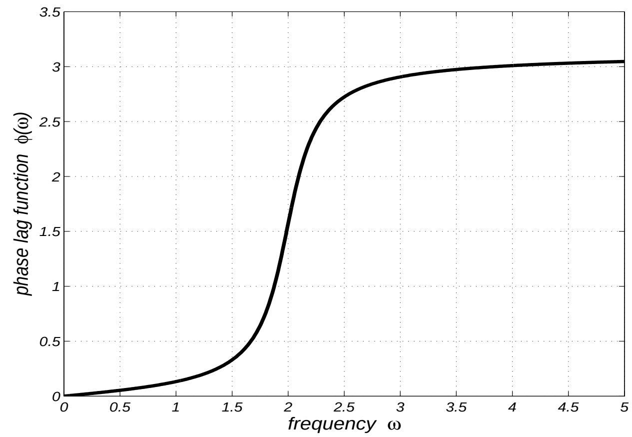
11. Consider the ODE

$$t^2\,x'' + 5t\,x' + 4x = 0.$$

Guessing $x(t) = t^\alpha$ will lead to a solution: $x_1(t)$. (Why is this a natural guess?) To find a second solution, $x_2(t)$, assume that $x_2(t) = u(t)\,x_1(t)$. Find an ODE that $u(t)$ must satisfy. Solve that ODE and find $x_2(t)$. Show that all initial value problems involving $t^2\,x'' + 5t\,x' + 4x = 0$ can be solved using a linear combination of $x_1(t)$ and $x_2(t)$.

12. Consider the forced, damped oscillator $y'' + 0.4\,y' + 4\,y = g(t)$. If the forcing is $g(t) = A\cos(\omega\,t)$ then the response is $Y_p = A\,|G(i\omega)|\,\cos(\omega\,t - \phi(\omega)) = A\,R(\omega)\,\cos(\omega\,t - \phi(\omega))$ where the graphs of the gain function $|G(i\omega)|$ (or $R(\omega)$) and the phase shift function $\phi(\omega)$ are shown below:





(a) Using these graphs, approximately what is the response to forcing $g(t) = a_1 \cos(t) + a_2 \cos(2t)$?

(b) Assume you have a forced, damped oscillator $y'' + 2\delta\, y' + \omega_0^2\, y = g(t)$ and a 2π periodic forcing $g(t)$ that is even about $t = 0$: $g(t) = a_0/2 + \sum a_n \cos(nt)$. How would you find the response to the forcing g? In your answer, just write things using $|G(i\omega)|$ and $\phi(\omega)$ (or $R(\omega)$ and $\phi(\omega)$) wherever you need.

(c) If you can measure the response to the forcing and if you know the parameters δ and ω_0, how can you use this information to determine the Fourier coefficients $\{a_n\}$ of the forcing function $g(t)$?

(d) Assume you have a black box that oscillates according to $a\, y'' + b\, y' + cy = 0$. ($a$, b, and c are positive.) What is the natural frequency ω_0 of the box?

(e) The box has a knob on it that allows you to change the value of c to be any positive number you want. You have a 2π periodic signal $g(t)$ that's even about $t = 0$. You would like to know a_7, the Fourier coefficient of $g(t)$ that multiplies $\cos(7t)$. How would you choose c so that you can compute a_7 from the response of the black box to this forcing?

The above problem is a rough cartoon of how an oscilloscope works; in an RLC circuit the coefficient c above is related to a capacitor. You can change the capacitance of a capacitor; see `https://en.wikipedia.org/wiki/Variable_capacitor`. In this way you can tune the circuit to different frequencies. And so you can construct a physical object that is tuned to specific frequencies. Physicists and engineers do this all the time.

Of course, you don't see many oscilloscopes anymore; nowadays what's done more often is that the signal is fed into a computer and the computer uses matlab (or whatever) to compute the Fourier transform and then it gives both a numeric result and to produce a display that looks like an oscilloscope (if you want). But before computers were small and common, people would solve ODEs by constructing electrical circuits whose behaviour was precisely the solution of the ODE in question. For example, see `https://en.wikibooks.org/wiki/Circuit_Theory/Analog_Computer` and `https://courses.engr.illinois.edu/ece486/fa2018/laboratory/docs/lab1/analog_computer_manual.pdf`

They still do this in devices where you need a fast result and you need the device to be a small (circuits are faster and smaller than computers) and they show up in controlled devices (devices whose behaviour is modified using control theory). This is a big enough market that matlab has an entire simulink package and there are people worldwide who think in terms of components of ciruits rather than terms in ODEs. `https://www.mathworks.com/products/simulink.html`. It's kind of mind-blowing if you think about it — there's a parallel world of people who build solutions to ODEs not by pen and paper but by soldering together the right components and they "speak component" as fluently as we speak mathematics.

If you find any of this at all intriguing, you might look into getting a Raspberry Pi[1]: a fully functioning computer the size of a cell phone that runs unix and can have a monitor and keyboards and speakers plugged into it. Or into getting an arduino[2]: a circuit board that you can program to do simple control theory. One of our math professors used one to replace the fan on his computer at home; he programmed it to turn on and off when it gets too hot. One of our chemistry professors used it to build a programmable thermostat for her parents' old furnace. There are Pi and arduino clubs on campus and around the city. And crash courses on them at the Toronto Public Library.

[1] `https://www.raspberrypi.org/`
[2] `https://www.arduino.cc/`