In general, there are some questions in the lecture notes that you should work on. Some come after proofs and some are in italics within proofs.

1. If y and z are two differentiable functions on (α, β) and $y(x) < z(x)$ for all $x \in (\alpha, \beta)$ does this imply that $y'(x) < z'(x)$ on (α, β)? Prove or disprove.

2. If $y'(x) < z'(x)$ for all $x \in (\alpha, \beta)$, does this imply that $y(x) < z(x)$ on (α, β)? Prove or disprove.

3. If $y'(x) < z'(x)$ for all $x \in \mathbb{R}$, does this imply that $y(x) < z(x)$ on $\mathbb{R}$? Prove or disprove.

4. If $y'(x) \le z'(x)$ for all $x \in \mathbb{R}$, does this imply that there can be at most one x at which $y(x) = z(x)$? Prove or disprove.

5. If $y'(x) < z'(x)$ for all $x \in \mathbb{R}$, does this imply that there can be at most one x at which $y(x) = z(x)$? Prove or disprove.

6. If $y''(x) < z''(x)$ for all $x \in \mathbb{R}$, does this imply that there can be at most two x at which $y(x) = z(x)$? Prove or disprove. If you disprove it, can you think of one more condition on the second derivatives so that it is true?

7. In the proof of Osgood's Uniqueness Theorem, why did we define $x_2 = \inf\{x < x_1 | z(x) > v(x)\}$ rather than simply saying "Let x_2 be some number between x_0 and x_1 such that $z(x_2) = v(x_2)$?

8. If there's an open set containing (x_0, y_0) where $f(x, y)$ is continuous in x and y and Lipschitz in y, then
$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$
has a unique solution in some open interval containing x_0. The solution has y' continuous on the open interval. What conditions would you need on f to know that y'' is also continuous? For y''' to be continuous?

9. So far, we've been thinking of solutions as single objects — as a graph or as a trajectory. Given an ODE we fix the initial data and think about the solution. This approach is useful in many ways but when geometers and dynamical systems folks are thinking about things, they often think in a more general manner. Please read the first page of Professor Yael Karshon's crash course on flows `http://www.math.toronto.edu/karshon/courses/symp/flows.pdf`. If you don't already know what a manifold is, you should by the end of MAT257. For the following, all you really need to know is what a diffeomorphism is.

   (a) Consider the interval $[0, 1]$ (this is our manifold M) and the vector field $f(y) = y(1 - y)$. The flow is $\phi_t(y_0) = y(t; y_0)$. That is $y(t; y_0)$ satisfies $y'(t; y_0) = f(y(t; y_0))$ and $y(0, y_0) = y_0$. (If the "$; y_0$)" notation is bugging you, $y(t; y_0)$ is the solution of $y' = f(y)$ such that $y(0) = y_0$.) You can exactly solve this ODE and so you can write down ϕ_t. Prove that it's a flow.

(b) More generally, we could take $M = \mathbb{R}$ or we could take $M = [a, b] \subset \mathbb{R}$ and then consider solutions of $y' = f(y)$ with initial data in M. We have our existence theorem and our existence and uniqueness theorem. What would we need to know about f so that we have a flow? If we're missing a theorem, what theorem are we missing?

(c) The definition of flow in Professor Karshon's notes requires that trajectories (aka solutions) exist for all time: $t \in \mathbb{R}$. We could relax this and think about flows as maps from $(\alpha, \beta) \times M \to M$ where (α, β) is an interval of existence that contains 0 and we require $\phi_{s+t}(m) = \phi_s(\phi_t(m))$ only when s, t, and $s + t$ are all in (a, b). With this relaxation, consider $M = [1, \infty)$ and the vector field $f(y) = y(1 - y)$. Again, you can solve this ODE and can write down $\phi_t(y)$. Is ϕ_t a flow? If not, why not? If not, how would your answer change if $M = [1, 100]$?

The following four pages of exercises are from Michael E. Taylor's *Introduction to Differential Equations.*

Exercises

1. Apply the Picard iteration method to

$$\frac{dx}{dt} = ax, \quad x(0) = 1,$$

given $a \in \mathbb{C}$. Taking $x_0(t) \equiv 1$, show that

$$x_n(t) = \sum_{k=0}^{n} \frac{a^k}{k!} t^k.$$

2. Discuss the matrix analogue of Exercise 1.

3. Consider the initial value problem

$$\frac{dx}{dt} = x^2, \quad x(0) = 1.$$

Take $x_0 \equiv 1$ and use the Picard iteration method (1.5) to write out

$$x_n(t), \quad n = 1, 2, 3.$$

Compare the results with the formula (1.23).

4. Given $A_0, A_1 \in M(n, \mathbb{C})$, consider the initial value problem

$$\frac{dx}{dt} = (A_0 + A_1 t)x, \quad x(0) = x_0.$$

Take $x_0(t) \equiv x_0$ and use the Picard iteration (1.5) to write out

$$x_n(t), \quad n = 1, 2, 3.$$

Compare and contrast the results with calculations from §10 of Chapter 3.

5. Modify the system (1.25) to

$$\frac{dy}{dt} = v, \quad \frac{dv}{dt} = -y^3 - v.$$

Show that solutions satisfy

$$\frac{d}{dt}\left(\frac{v^2}{2} + \frac{y^4}{4}\right) \le 0,$$

and use this to establish global existence for $t \geq 0$.

6. Consider the initial value problem

$$\frac{dx}{dt} = |x|^{1/2}, \quad x(0) = 0.$$

Note that $x(t) \equiv 0$ is a solution, and

$$x(t) = \begin{cases} \frac{1}{4}t^2, & t \geq 0, \\ 0, & t \leq 0 \end{cases}$$

is another solution, on $t \in (-\infty, \infty)$. Why does this not contradict the uniqueness part of Proposition 1.1? Can you produce other solutions to this initial value problem?

7. Take $\beta \in (0, \infty)$ and consider the initial value problem

$$\frac{dx}{dt} = x^\beta, \quad x(0) = 1.$$

Show that this has a solution for all $t \geq 0$ if and only if $\beta \leq 1$.

8. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 and suppose $x(t)$ solves

$$\frac{dx}{dt} = F(x), \quad x(t_0) = x_0, \tag{1.43}$$

for $t \in I$, an open interval containing t_0. Show that, for $t \in I$,

$$\frac{d}{dt}\|x(t)\|^2 = 2x(t) \cdot F(x(t)). \tag{1.44}$$

Show that, if $\alpha > 0$ and $x(t) \neq 0$,

$$\frac{d}{dt}\|x(t)\|^\alpha = \alpha\|x(t)\|^{\alpha-2}x(t) \cdot F(x(t)). \tag{1.45}$$

9. In the setting of Exercise 8, suppose F satisfies an estimate

$$\|F(x)\| \leq C(1 + \|x\|)^\beta, \quad \forall x \in \mathbb{R}^n, \quad C < \infty, \quad \beta < 1. \tag{1.46}$$

Show that there exist $\alpha > 0$ and $K < \infty$ such that, if $\|x(t)\| \geq 1$ for $t \in I$,

$$\frac{d}{dt}\|x(t)\|^\alpha \leq K, \quad \forall t \in I.$$

Use this to establish that the solution to (1.43) exists for all $t \in \mathbb{R}$.

Exercises 10–12 below will extend the conclusion of Exercise 9 to the case $\beta = 1$ in (1.46). One approach is via the following result, known as *Gronwall's inequality.*

Proposition 1.5. *Assume*

$$g \in C^1(\mathbb{R}), \quad g' \geq 0. \tag{1.47}$$

Let u and v be real-valued, continuous functions on I satisfying

$$\begin{aligned} u(t) &\leq A + \int_{t_0}^{t} g(u(s))\, ds, \\ v(t) &\geq A + \int_{t_0}^{t} g(v(s))\, ds. \end{aligned} \tag{1.48}$$

Then

$$u(t) \leq v(t), \quad \text{for } t \in I,\ t \geq t_0. \tag{1.49}$$

Proof. Set $w(t) = u(t) - v(t)$. Then

$$\begin{aligned} w(t) &\leq \int_{t_0}^{t} \big[g(u(s)) - g(v(s))\big]\, ds \\ &= \int_{t_0}^{t} M(s) w(s)\, ds, \end{aligned} \tag{1.50}$$

where

$$M(s) = \int_0^1 g'(\tau u(s) + (1-\tau) v(s))\, d\tau. \tag{1.51}$$

Hence we have

$$w(t) \leq \int_{t_0}^{t} M(s) w(s)\, ds, \quad M(s) \geq 0, \quad M \in C(I), \tag{1.52}$$

and we claim that this implies

$$w(t) \leq 0, \quad \forall t \in I,\ t \geq t_0. \tag{1.53}$$

In other words, we claim that $w(t) \leq 0$ on $[t_0, b]$ whenever $[t_0, b] \subset I$. To see this, let t_1 be the largest number in $[t_0, b]$ with the property that $w \leq 0$ on $[t_0, t_1]$. We claim that $t_1 = b$.

Assume to the contrary that $t_1 < b$. Noting that $\int_{t_0}^{t_1} M(s)w(s)\,ds \le 0$, we deduce from (1.52) that

(1.54) $$w(t) \le \int_{t_1}^{t} M(s)w(s)\,ds, \quad \forall\, t \in [t_1, b].$$

Hence, with

(1.56) $$K = \max_{[t_1,b]} M(s) < \infty,$$

we have, for $a \in (t_1, b)$,

(1.57) $$\max_{[t_1,a]} w(t) \le (a - t_1)K \max_{[t_1,a]} w(s).$$

If we pick $a \in (t_1, b)$ such that $(a - t_1)K < 1$, this implies

(1.58) $$w(t) \le 0, \quad \forall\, t \in [t_1, a],$$

contradicting the maximality of t_1. Hence actually $t_1 = b$, and we have the implication (1.52) $\Rightarrow$ (1.53), completing the proof of Proposition 1.5.

10. Assume $v \ge 0$ is a C^1 function on $I = (a, b)$, satisfying

(1.59) $$\frac{dv}{dt} \le Cv, \quad v(t_0) = v_0,$$

where $C \in (0, \infty)$ and $t_0 \in I$. Using Proposition 1.5, show that

(1.60) $$v(t) \le e^{C(t-t_0)}v_0, \quad \forall\, t \in [t_0, b).$$

11. In the setting of Exercise 10, avoid use of Proposition 1.5 as follows. Write (1.59) as

(1.61) $$\frac{dv}{dt} = Cv - g(t), \quad v(t_0) = v_0, \quad g \ge 0,$$

with solution

(1.62) $$v(t) = e^{C(t-t_0)}v_0 - \int_{t_0}^{t} e^{C(t-s)}g(s)\,ds.$$

Deduce (1.60) from this.

12. Return to the setting of Exercise 8, and replace the hypothesis (1.46) by

(1.63) $$\|F(x)\| \le C(1 + \|x\|), \quad \forall\, x \in \mathbb{R}^n.$$

Show that the solution to (1.43) exists for all $t \in \mathbb{R}$.
Hint. Take $v(t) = 1 + \|x(t)\|^2$ and use (1.44). Show that Exercise 10 (or 11) applies.