"Differential Equations: an Introduction to Modern Methods + Applications" J.R. Brannan W.E. Boyce

In each of Problems 54 through 61, find the general solution of the given Cauchy–Euler equation in $x > 0$:

54. $x^2y'' + xy' + 4y = 0$

55. $x^2y'' + 4xy' + 2y = 0$

56. $x^2y'' + 3xy' + 1.25y = 0$

57. $x^2y'' - 4xy' - 6y = 0$

58. $x^2y'' - 2y = 0$

59. $x^2y'' - 5xy' + 9y = 0$

60. $x^2y'' + 2xy' + 4y = 0$

61. $2x^2y'' - 4xy' + 6y = 0$

In each of Problems 62 through 65, find the solution of the given initial value problem. Plot the graph of the solution and describe how the solution behaves as $x \to 0$.

62. $2x^2y'' + xy' - 3y = 0, \quad y(1) = 1, \quad y'(1) = 1$

63. $4x^2y'' + 8xy' + 17y = 0, \quad y(1) = 2, \quad y'(1) = -3$

64. $x^2y'' - 5xy' + 9y = 0, \quad y(-1) = 2, \quad y'(-1) = 3$

65. $x^2y'' + 3xy' + 5y = 0, \quad y(1) = 1, \quad y'(1) = -1$

4.4 Mechanical and Electrical Vibrations

In Section 4.1 the mathematical models derived for the spring-mass system, the linearized pendulum, and the RLC circuit all turned out to be linear constant coefficient differential equations that, in the absence of a forcing function, are of the form

$$ay'' + by' + cy = 0. \tag{1}$$

To adapt Eq. (1) to a specific application merely requires interpretation of the coefficients in terms of the physical parameters that characterize the application. Using the theory and methods developed in Sections 4.2 and 4.3, we are able to solve Eq. (1) completely for all possible parameter values and initial conditions. Thus Eq. (1) provides us with an important class of problems that illustrates the linear theory described in Section 4.2 and solution methods developed in Section 4.3.

Undamped Free Vibrations

Recall that the equation of motion for the damped spring-mass system with external forcing is

$$my'' + \gamma y' + ky = F(t). \tag{2}$$

Equation (2) and the pair of conditions,

$$y(0) = y_0, \qquad y'(0) = v_0, \tag{3}$$

that specify initial position y_0 and initial velocity v_0 provide a complete formulation of the vibration problem. If there is no external force, then $F(t) = 0$ in Eq. (2).

Let us also suppose that there is no damping, so that $\gamma = 0$. This is an idealized configuration of the system, seldom (if ever) completely attainable in practice. However, if the actual damping is very small, then the assumption of no damping may yield satisfactory results over short to moderate time intervals. In this case, the equation of motion (2) reduces to

$$my'' + ky = 0. \tag{4}$$

If we divide Eq. (4) by m, it becomes

$$y'' + \omega_0^2 y = 0, \tag{5}$$

where

$$\omega_0^2 = k/m. \tag{6}$$

The characteristic equation for Eq. (5) is

$$\lambda^2 + \omega_0^2 = 0, \tag{7}$$

and the corresponding characteristic roots are $\lambda = \pm i\,\omega_0$. It follows that the general solution of Eq. (5) is

$$y = A\cos\omega_0 t + B\sin\omega_0 t. \tag{8}$$

where A and B are arbitrary constants. Substituting from Eq. (8) into the initial conditions (3) determines the integration constants A and B in terms of initial position and velocity, $A = y_0$ and $B = v_0/\omega_0$.

In discussing the solution of Eq. (5), it is convenient to rewrite Eq. (8) in the **phase-amplitude** form

$$y = R\cos(\omega_0 t - \delta). \tag{9}$$

To see the relationship between Eqs. (8) and (9), use the trigonometric identity for the cosine of the difference of the two angles, $\omega_0 t$ and δ, to rewrite Eq. (9) as

$$y = R\cos\delta\cos\omega_0 t + R\sin\delta\sin\omega_0 t. \tag{10}$$

By comparing Eq. (10) with Eq. (8), we find that A, B, R, and δ are related by the equations

$$A = R\cos\delta, \qquad B = R\sin\delta. \tag{11}$$

From these two equations, we see that (R, δ) is simply the polar coordinate representation of the point with Cartesian coordinates (A, B) (Figure 4.4.1).

Thus

$$R = \sqrt{A^2 + B^2}, \tag{12}$$

while δ satisfies

$$\cos\delta = \frac{A}{\sqrt{A^2 + B^2}}, \qquad \sin\delta = \frac{B}{\sqrt{A^2 + B^2}}. \tag{13}$$

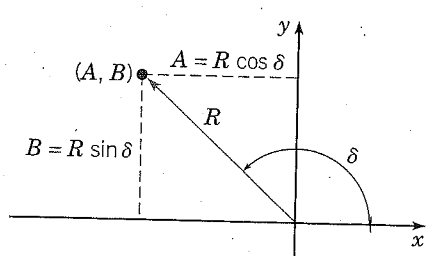

FIGURE 4.4.1 Relation between (R, δ) in Eq. (9) and (A, B) in Eq. (8).

Let arctan (B/A) be the angle that lies in the principal branch of the inverse tangent function, that is, in the interval $-\pi/2 < \hat{\delta} < \pi/2$ (Figure 4.4.2). Then the values of δ given by

$$\delta = \begin{cases} \arctan(B/A), & \text{if } A > 0, B \geq 0 \text{ (1st quadrant)} \\ \pi + \arctan(B/A), & \text{if } A < 0 \text{ (2nd or 3rd quadrant)} \\ 2\pi + \arctan(B/A), & \text{if } A > 0, B < 0 \text{ (4th quadrant)} \\ \pi/2, & \text{if } A = 0, B > 0 \\ 3\pi/2, & \text{if } A = 0, B < 0 \end{cases}$$

will lie in the interval $[0, 2\pi)$.

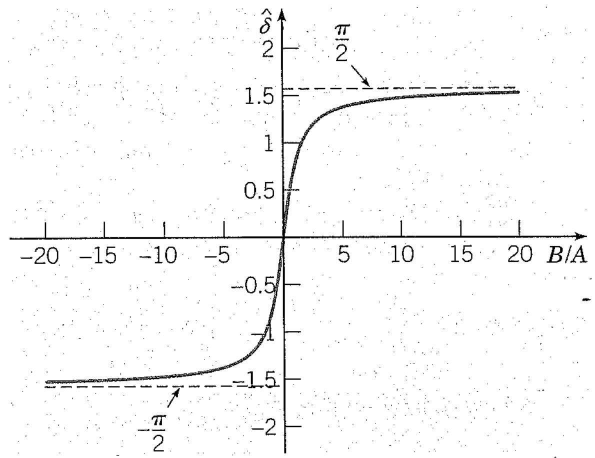

FIGURE 4.4.2 The principal branch of the arctangent function.

The graph of Eq. (9), or the equivalent Eq. (8), for a typical set of initial conditions is shown in Figure 4.4.3. The graph is a displaced cosine wave that describes a periodic, or simple harmonic, motion of the mass. The **period** of the motion is

$$T = \frac{2\pi}{\omega_0} = 2\pi\left(\frac{m}{k}\right)^{1/2}. \tag{14}$$

The circular frequency $\omega_0 = \sqrt{k/m}$, measured in radians per unit time, is called the **natural frequency** of the vibration. The maximum displacement R of the mass from equilibrium is the **amplitude** of the motion. The dimensionless parameter δ is called the **phase**, or phase

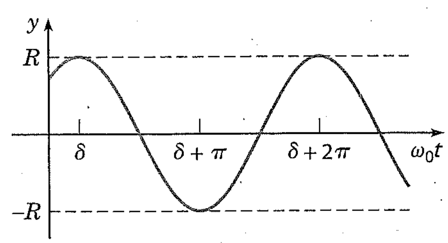

FIGURE 4.4.3 Simple harmonic motion $y = R\cos(\omega_0 t - \delta)$.

angle. The quantity δ/ω_0 measures the time shift of the wave from its normal position corresponding to $\delta = 0$.

Note that the motion described by Eq. (9) has a constant amplitude that does not diminish with time. This reflects the fact that, in the absence of damping, there is no way for the system to dissipate the energy imparted to it by the initial displacement and velocity. Further, for a given mass m and spring constant k, the system always vibrates at the same frequency ω_0, regardless of the initial conditions. However the initial conditions do help to determine the amplitude of the motion. Finally, observe from Eq. (14) that T increases as m increases, so larger masses vibrate more slowly. On the other hand, T decreases as k increases, which means that stiffer springs cause the system to vibrate more rapidly.

EXAMPLE 1

Suppose that a mass weighing 10 lb stretches a spring 2 in. If the mass is displaced an additional 2 in. and is then set in motion with an initial upward velocity of 1 ft/s, determine the position of the mass at any later time. Also determine the period, amplitude, and phase of the motion.

The spring constant is $k = 10$ lb/2 in. $= 60$ lb/ft, and the mass is $m = w/g = \frac{10}{32}$ lb·s^2/ft. Hence the equation of motion reduces to

$$y'' + 192y = 0, \tag{15}$$

and the general solution is

$$y = A\cos(8\sqrt{3}t) + B\sin(8\sqrt{3}t).$$

The solution satisfying the initial conditions $y(0) = \frac{1}{6}$ ft and $y'(0) = -1$ ft/s is

$$y = \frac{1}{6}\cos(8\sqrt{3}t) - \frac{1}{8\sqrt{3}}\sin(8\sqrt{3}t), \tag{16}$$

that is, $A = \frac{1}{6}$ and $B = -1/(8\sqrt{3})$. The natural frequency is $\omega_0 = \sqrt{192} \cong 13.856$ radians (rad)/s, so the period is $T = 2\pi/\omega_0 \cong 0.45345$ s. The amplitude R and phase δ are found from Eqs. (12) and (13). We have

$$R^2 = \frac{1}{36} + \frac{1}{192} = \frac{19}{576}, \quad \text{so} \quad R \cong 0.18162 \text{ ft}.$$

and since $A > 0$ and $B < 0$, the angle δ lies in the fourth quadrant,

$$\delta = 2\pi + \arctan(-\sqrt{3}/4) \cong 5.87455 \text{ rad}.$$

The graph of the solution (16) is shown in Figure 4.4.4.

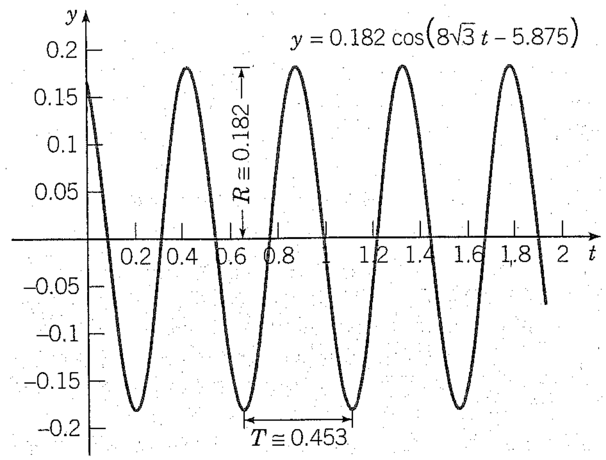

FIGURE 4.4.4 An undamped free vibration: $y'' + 192y = 0$, $y(0) = \frac{1}{6}$, $y'(0) = -1$.

Damped Free Vibrations

If we include the effect of damping, the differential equation governing the motion of the mass is

$$my'' + \gamma y' + ky = 0. \tag{17}$$

We are especially interested in examining the effect of variations in the damping coefficient γ for given values of the mass m and spring constant k. The roots of the corresponding characteristic equation,

$$m\lambda^2 + \gamma\lambda + k = 0, \tag{18}$$

are

$$\lambda_1, \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right). \tag{19}$$

There are three cases to consider, depending on the sign of the discriminant $\gamma^2 - 4km$.

1. **Underdamped Harmonic Motion** ($\gamma^2 - 4km < 0$). In this case, the roots in Eq. (19) are complex numbers $\mu \pm i\nu$ with $\mu = -\gamma/2m < 0$ and $\nu = \dfrac{(4km - \gamma^2)^{1/2}}{2m} > 0$. Hence the general solution of Eq. (18) is

$$y = e^{-\gamma t/2m}(A\cos \nu t + B\sin \nu t). \tag{20}$$

2. **Critically Damped Harmonic Motion** ($\gamma^2 - 4km = 0$). In this case, $\lambda_1 = -\gamma/2m < 0$ is a repeated root. Therefore the general solution of Eq. (17) in this case is

$$y = (A + Bt)e^{-\gamma t/2m}. \tag{21}$$

3. Overdamped Harmonic Motion ($\gamma^2 - 4km > 0$). Since m, γ, and k are positive, $\gamma^2 - 4km$ is always less than γ^2. In this case, the values of λ_1 and λ_2 given by Eq. (19) are real, distinct, and *negative*, and the general solution of Eq. (17) is

$$y = Ae^{\lambda_1 t} + Be^{\lambda_2 t}. \tag{22}$$

Since the roots in Eq. (19) are either real and negative, or complex with a negative real part, in all cases the solution y tends to zero as $t \to \infty$; this occurs regardless of the values of the arbitrary constants A and B, that is, regardless of the initial conditions. This confirms our intuitive expectation, namely, that damping gradually dissipates the energy initially imparted to the system, and consequently, the motion dies out with increasing time.

The most important case is the first one, which occurs when the damping is small. If we let $A = R\cos\delta$ and $B = R\sin\delta$ in Eq. (20), then we obtain

$$y = Re^{-\gamma t/2m}\cos(\nu t - \delta). \tag{23}$$

The displacement y lies between the curves $y = \pm Re^{-\gamma t/2m}$; hence it resembles a cosine wave whose amplitude decreases as t increases. A typical example is sketched in Figure 4.4.5. The motion is called a damped oscillation or damped vibration. The amplitude factor R depends on m, γ, k, and the initial conditions.

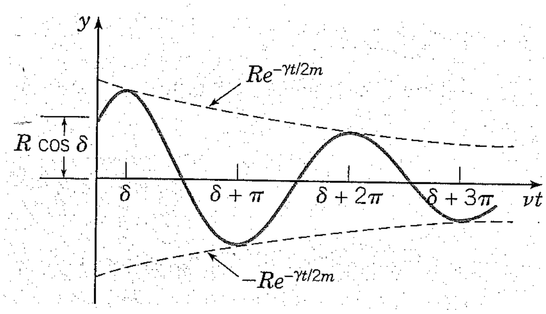

FIGURE 4.4.5 Damped vibration; $y = Re^{-\gamma t/2m}\cos(\nu t - \delta)$.

Although the motion is not periodic, the parameter ν determines the frequency with which the mass oscillates back and forth; consequently, ν is called the **quasi-frequency**. By comparing ν with the frequency ω_0 of undamped motion, we find that

$$\frac{\nu}{\omega_0} = \frac{(4km - \gamma^2)^{1/2}/2m}{\sqrt{k/m}} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2} \cong 1 - \frac{\gamma^2}{8km}. \tag{24}$$

The last approximation is valid when $\gamma^2/4km$ is small. We refer to this situation as "small damping." Thus the effect of small damping is to reduce slightly the frequency of the oscillation. By analogy with Eq. (14), the quantity $T_d = 2\pi/\nu$ is called the **quasi-period**. It is the time between successive maxima or successive minima of the position of the mass, or between successive passages of the mass through its equilibrium position while going in the same direction. The relation between T_d and T is given by

$$\frac{T_d}{T} = \frac{\omega_0}{\nu} = \left(1 - \frac{\gamma^2}{4km}\right)^{-1/2} \cong 1 + \frac{\gamma^2}{8km}, \tag{25}$$

where again the last approximation is valid when $\gamma^2/4km$ is small. Thus small damping increases the quasi-period.

Equations (24) and (25) reinforce the significance of the dimensionless ratio $\gamma^2/4km$. It is not the magnitude of γ alone that determines whether damping is large or small, but the magnitude of γ^2 compared to $4km$. When $\gamma^2/4km$ is small, then damping has a small effect on the quasi-frequency and quasi-period of the motion. On the other hand, if we want to study the detailed motion of the mass for all time, then we can *never* neglect the damping force, no matter how small.

As $\gamma^2/4km$ increases, the quasi-frequency ν decreases and the quasi-period T_d increases. In fact, $\nu \to 0$ and $T_d \to \infty$ as $\gamma \to 2\sqrt{km}$. As indicated by Eqs. (20), (21), and (22), the nature of the solution changes as γ passes through the value $2\sqrt{km}$. This value of γ is known as critical damping. The motion is said to be underdamped for values of $\gamma < 2\sqrt{km}$; while for values of $\gamma > 2\sqrt{km}$, the motion is said to be overdamped. In the critically damped and overdamped cases given by Eqs. (21) and (20), respectively, the mass creeps back to its equilibrium position but does not oscillate about it, as for small γ. Note that this analysis is consistent with the definitions of underdamped, critcally damped, and overdamped harmonic motion based on the sign of $\gamma^2 - 4km$ (see pages 245–246). Two typical examples of critically damped motion are shown in Figure 4.4.6, and the situation is discussed further in Problems 19 and 20.

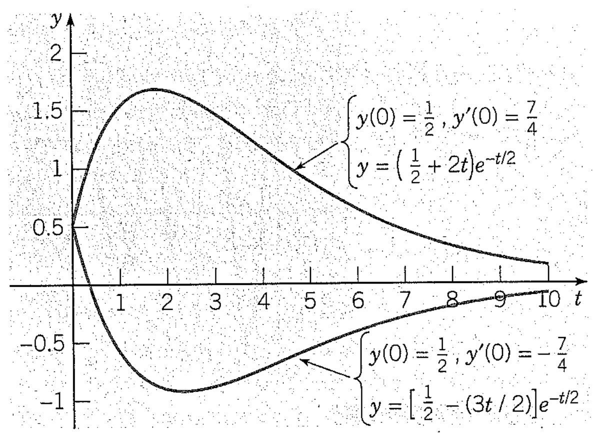

FIGURE 4.4.6 Two critically damped motions: $y'' + y' + 0.25y = 0$; $y = (A + Bt)e^{-t/2}$.

EXAMPLE 2

The motion of a certain spring-mass system is governed by the differential equation

$$y'' + 0.125y' + y = 0, \tag{26}$$

where y is measured in feet and t in seconds. If $y(0) = 2$ and $y'(0) = 0$, determine the position of the mass at any time. Find the quasi-frequency and the quasi-period, as well as the time at which the mass first passes through its equilibrium position. Find the time τ such that $|y(t)| < 0.1$ for all $t > \tau$.

The solution of Eq. (26) is

$$y = e^{-t/16}\left[A\cos\frac{\sqrt{255}}{16}t + B\sin\frac{\sqrt{255}}{16}t\right].$$

To satisfy the initial conditions, we must choose $A = 2$ and $B = 2/\sqrt{255}$; hence the solution of the initial value problem is

$$\begin{aligned} y &= e^{-t/16}\left(2\cos\frac{\sqrt{255}}{16}t + \frac{2}{\sqrt{255}}\sin\frac{\sqrt{255}}{16}t\right) \\ &= \frac{32}{\sqrt{255}}e^{-t/16}\cos\left(\frac{\sqrt{255}}{16}t - \delta\right), \end{aligned} \tag{27}$$

where $\tan\delta = 1/\sqrt{255}$, so $\delta \cong 0.06254$. The displacement of the mass as a function of time is shown in Figure 4.4.7. For purposes of comparison, we also show the motion if the damping term is neglected.

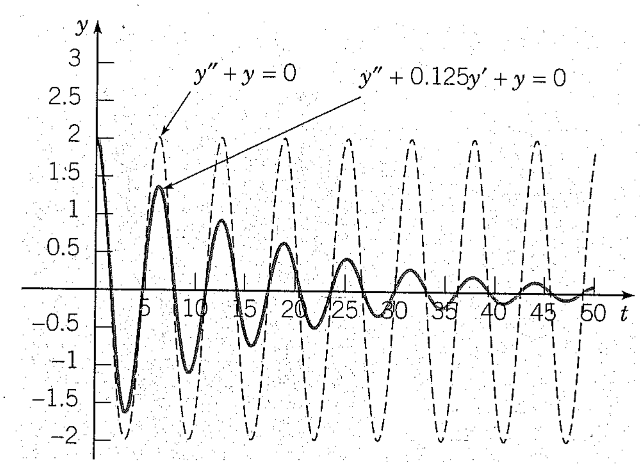

FIGURE 4.4.7 Vibration with small damping (solid curve) and with no damping (dashed curve). In each case, $y(0) = 2$ and $y'(0) = 0$.

The quasi-frequency is $\nu = \sqrt{255}/16 \cong 0.998$ and the quasi-period is $T_d = 2\pi/\nu \cong 6.295$ s. These values differ only slightly from the corresponding values (1 and 2π, respectively) for the undamped oscillation. This is also evident from the graphs in Figure 4.4.7, which rise and fall almost together. The damping coefficient is small in this example, only one-sixteenth of the critical value, in fact. Nevertheless the amplitude of the oscillation is reduced rather rapidly. Figure 4.4.8 shows the graph of the solution for $40 \le t \le 60$, together with the graphs of $y = \pm 0.1$. From the graph, it appears that τ is about 47.5, and by a more precise calculation we find that $\tau \cong 47.5149$ s.

To find the time at which the mass first passes through its equilibrium position, we refer to Eq. (27) and set $\sqrt{255}t/16 - \delta$ equal to $\pi/2$, the smallest positive zero of the cosine function. Then, by solving for t, we obtain

$$t = \frac{16}{\sqrt{255}}\left(\frac{\pi}{2} + \delta\right) \cong 1.637 \text{ s}.$$

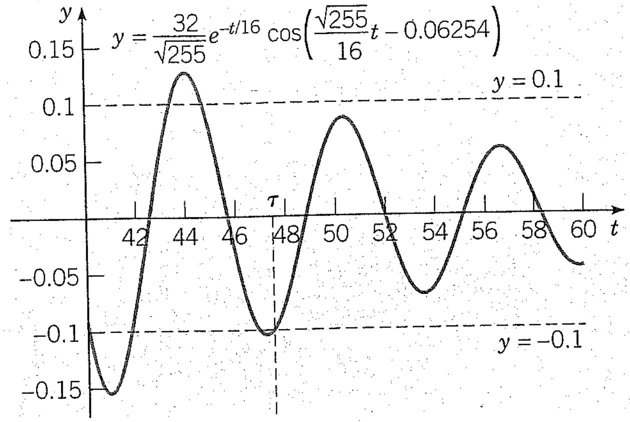

FIGURE 4.4.8 Solution of Example 2; determination of τ.

Phase Portraits for Harmonic Oscillators

The differences in the behavior of solutions of undamped and damped harmonic oscillators, illustrated by plots of displacement versus time, are completed by looking at their corresponding phase portraits. If we convert Eq. (17) to a first order system where $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} = y\mathbf{i} + y'\mathbf{j}$, we obtain

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 0 & 1 \\ -k/m & -\gamma/m \end{pmatrix}\mathbf{x}. \tag{28}$$

Since the eigenvalues of $\mathbf{A}$ are the roots of the characteristic equation (18), we know that the origin of the phase plane is a center, and therefore stable, for the undamped system in which $\gamma = 0$. In the underdamped case, $0 < \gamma^2 < 4km$, the origin is a spiral sink. Direction fields and phase portraits for these two cases are shown in Figure 4.4.9.

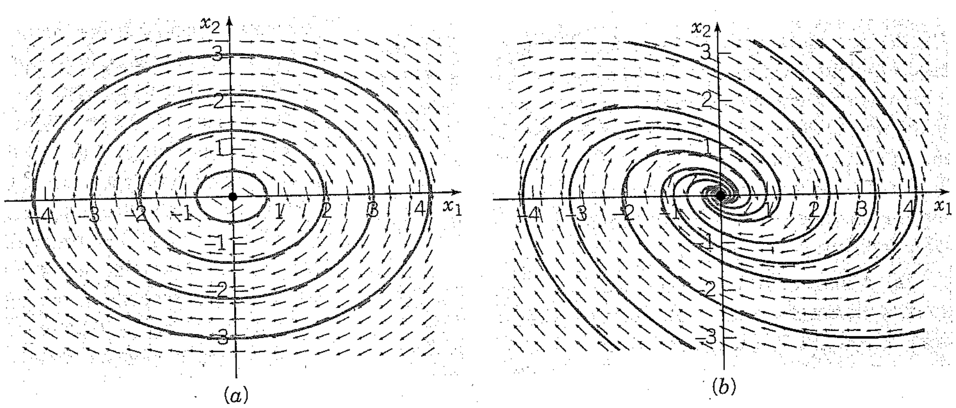

FIGURE 4.4.9 Direction field and phase portrait for (*a*) an undamped harmonic oscillator. (*b*) a damped harmonic oscillator that is underdamped.

If $\gamma_2 = 4km$, the matrix **A** has a negative, real, and repeated eigenvalue; if $\gamma^2 > 4km$, the eigenvalues of **A** are real, negative, and unequal. Thus the origin of the phase plane in both the critically damped and overdamped cases is a nodal sink. Direction fields and phase portraits for these two cases are shown in Figure 4.4.10.

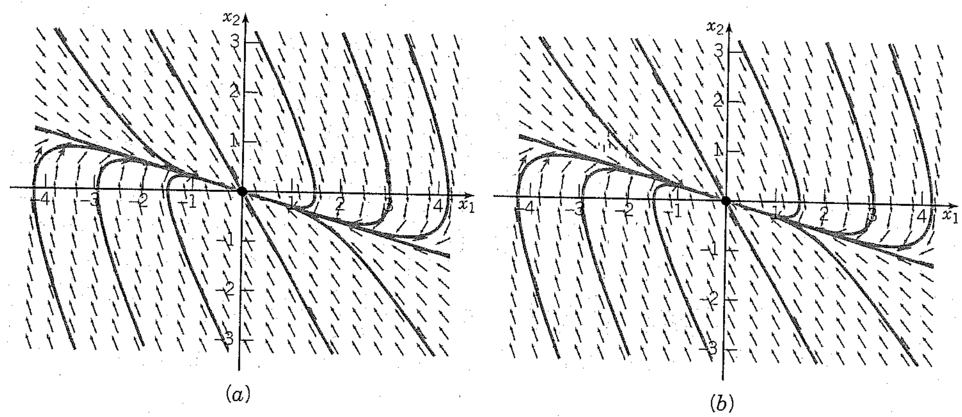

FIGURE 4.4.10 Direction field and phase portrait for (a) a critically damped harmonic oscillator. (b) an overdamped harmonic oscillator.

It is clear from the phase portraits in Figure 4.4.10 that a mass can pass through the equilibrium position at most once, since trajectories either do not cross the x_2-axis, or cross it at most once, as they approach the equilibrium point. In Problem 19, you are asked to give an analytic argument of this fact.

PROBLEMS

In each of Problems 1 through 4, determine ω_0, R, and δ so as to write the given expression in the form $y = R\cos(\omega_0 t - \delta)$.

1. $y = 3\cos 2t + 3\sin 2t$
2. $y = -\cos t + \sqrt{3}\sin t$
3. $y = 4\cos 3t - 2\sin 3t$
4. $y = -2\sqrt{3}\cos \pi t - 2\sin \pi t$

5. **(a)** A mass weighing 2 lb stretches a spring 6 in. If the mass is pulled down an additional 3 in. and then released, and if there is no damping, determine the position y of the mass at any time t. Plot y versus t. Find the frequency, period, and amplitude of the motion.
(b) Draw a phase portrait of the equivalent dynamical system that includes the trajectory corresponding to the initial value problem in part (a).

6. **(a)** A mass of 100 g stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/s, and if there is no damping, determine the position y of the mass at any time t. When does the mass first return to its equilibrium position?
(b) Draw a phase portrait of the equivalent dynamical system that includes the trajectory corresponding to the initial value problem in part (a).

7. A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in., and then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position y of the mass at any time t. Determine the frequency, period, amplitude, and phase of the motion.

8. A series circuit has a capacitor of 0.25 microfarad and an inductor of 1 henry. If the initial charge on the capacitor is 10^{-6} coulomb and there is no initial current, find the charge q on the capacitor at any time t.

9. **(a)** A mass of 20 g stretches a spring 5 cm. Suppose that the mass is also attached to a viscous damper with a damping constant of 400 dyne·s/cm. If the mass is pulled down an additional 2 cm and then released, find its position y at any time t. Plot y versus t. Determine the quasi-frequency and the quasi-period. Determine the ratio of the quasi-period to the period of the corresponding undamped motion. Also find the time τ such that $|y(t)| < 0.05$ cm for all $t > \tau$.
(b) Draw a phase portrait of the equivalent dynamical system that includes the trajectory corresponding to the initial value problem in part (a).

10. A mass weighing 16 lb stretches a spring 3 in. The mass is attached to a viscous damper with a damping constant of

2 lb·s/ft. If the mass is set in motion from its equilibrium position with a downward velocity of 3 in./s, find its position y at any time t. Plot y versus t. Determine when the mass first returns to its equilibrium position. Also find the time τ such that $|y(t)| < 0.01$ in for all $t > \tau$.

11. **(a)** A spring is stretched 10 cm by a force of 3 newtons (N). A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass is 5 m/s. If the mass is pulled down 5 cm below its equilibrium position and given an initial downward velocity of 10 cm/s, determine its position y at any time t. Find the quasi-frequency ν and the ratio of ν to the natural frequency of the corresponding undamped motion.
(b) Draw a phase portrait of the equivalent dynamical system that includes the trajectory corresponding to the initial value problem in part (a).

12. **(a)** A series circuit has a capacitor of 10^{-5} farad, a resistor of 3×10^2 ohms, and an inductor of 0.2 henry. The initial charge on the capacitor is 10^{-6} coulomb and there is no initial current. Find the charge q on the capacitor at any time t.
(b) Draw a phase portrait of the equivalent dynamical system that includes the trajectory corresponding to the initial value problem in part (a).

13. A certain vibrating system satisfies the equation $y'' + \gamma y' + y = 0$. Find the value of the damping coefficient γ for which the quasi-period of the damped motion is 50% greater than the period of the corresponding undamped motion.

14. Show that the period of motion of an undamped vibration of a mass hanging from a vertical spring is $2\pi\sqrt{L/g}$, where L is the elongation of the spring due to the mass and g is the acceleration due to gravity.

15. Show that the solution of the initial value problem

$$my'' + \gamma y' + ky = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1$$

can be expressed as the sum $y = v + w$, where v satisfies the initial conditions $v(t_0) = y_0$, $v'(t_0) = 0$, w satisfies the initial conditions $w(t_0) = 0$, $w'(t_0) = y_1$, and both v and w satisfy the same differential equation as u. This is another instance of superposing solutions of simpler problems to obtain the solution of a more general problem.

16. Show that $A\cos\omega_0 t + B\sin\omega_0 t$ can be written in the form $r\sin(\omega_0 t - \theta)$. Determine r and θ in terms of A and B. If $R\cos(\omega_0 t - \delta) = r\sin(\omega_0 t - \theta)$, determine the relationship among R, r, δ, and θ.

17. A mass weighing 8 lb stretches a spring 1.5 in. The mass is also attached to a damper with coefficient γ. Determine the value of γ for which the system is critically damped. Be sure to give the units for γ.

18. If a series circuit has a capacitor of $C = 0.8$ microfarad and an inductor of $L = 0.2$ henry, find the resistance R so that the circuit is critically damped.

19. Assume that the system described by the equation $my'' + \gamma y' + ky = 0$ is either critically damped or overdamped. Show that the mass can pass through the equilibrium position at most once, regardless of the initial conditions.
Hint: Determine all possible values of t for which $y = 0$.

20. Assume that the system described by the equation $my'' + \gamma y' + ky = 0$ is critically damped and that the initial conditions are $y(0) = y_0, y'(0) = v_0$. If $v_0 = 0$, show that $y \to 0$ as $t \to \infty$ but that y is never zero. If y_0 is positive, determine a condition on v_0 that will ensure that the mass passes through its equilibrium position after it is released.

21. **Logarithmic Decrement**
(a) For the damped oscillation described by Eq. (23), show that the time between successive maxima is $T_d = 2\pi/\nu$.
(b) Show that the ratio of the displacements at two successive maxima is given by $\exp(\gamma T_d/2m)$. Observe that this ratio does not depend on which pair of maxima is chosen. The natural logarithm of this ratio is called the logarithmic decrement and is denoted by Δ.
(c) Show that $\Delta = \pi\gamma/m\nu$. Since m, ν, and Δ are quantities that can be measured easily for a mechanical system, this result provides a convenient and *practical* method for determining the damping constant of the system, which is more difficult to measure directly. In particular, for the motion of a vibrating mass in a viscous fluid, the damping constant depends on the viscosity of the fluid. For simple geometric shapes, the form of this dependence is known, and the preceding relation allows the experimental determination of the viscosity. This is one of the most accurate ways of determining the viscosity of a gas at high pressure.

22. Referring to Problem 21, find the logarithmic decrement of the system in Problem 10.

23. For the system in Problem 17, suppose that $\Delta = 3$ and $T_d = 0.3$ s. Referring to Problem 21, determine the value of the damping coefficient γ.

24. The position of a certain spring-mass system satisfies the initial value problem

$$\tfrac{3}{2}y'' + ky = 0, \qquad y(0) = 2, \quad y'(0) = v.$$

If the period and amplitude of the resulting motion are observed to be π and 3, respectively, determine the values of k and v.

25. Consider the initial value problem

$$y'' + \gamma y' + y = 0, \qquad y(0) = 2, \quad y'(0) = 0.$$

We wish to explore how long a time interval is required for the solution to become "negligible" and how this interval depends on the damping coefficient γ. To be more precise, let us seek the time τ such that $|y(t)| < 0.01$ for all $t > \tau$. Note that critical damping for this problem occurs for $\gamma = 2$.

(a) Let $\gamma = 0.25$ and determine τ, or at least estimate it fairly accurately from a plot of the solution.
(b) Repeat part (a) for several other values of γ in the interval $0 < \gamma < 1.5$. Note that τ steadily decreases as γ increases for γ in this range.
(c) Create a graph of τ versus γ by plotting the pairs of values found in parts (a) and (b). Is the graph a smooth curve?
(d) Repeat part (b) for values of γ between 1.5 and 2. Show that τ continues to decrease until γ reaches a certain critical value γ_0, after which τ increases. Find γ_0 and the corresponding minimum value of τ to two decimal places.
(e) Another way to proceed is to write the solution of the initial value problem in the form (23). Neglect the cosine factor and consider only the exponential factor and the amplitude R. Then find an expression for τ as a function of γ. Compare the approximate results obtained in this way with the values determined in parts (a), (b), and (d).

26. Consider the initial value problem

$$my'' + \gamma y' + ky = 0, \quad y(0) = y_0, \quad y'(0) = v_0.$$

Assume that $\gamma^2 < 4km$.
(a) Solve the initial value problem.
(b) Write the solution in the form $y(t) = R\exp(-\gamma t/2m)\cos(\nu t - \delta)$. Determine R in terms of m, γ, k, y_0, and v_0.
(c) Investigate the dependence of R on the damping coefficient γ for fixed values of the other parameters.

27. Use the differential equation derived in Problem 19 of Section 4.1 to determine the period of vertical oscillations of a cubic block floating in a fluid under the stated conditions.

28. Draw the phase portrait for the dynamical system equivalent to the differential equation considered in Example 2: $y'' + 0.125y' + y = 0$.

29. The position of a certain undamped spring-mass system satisfies the initial value problem

$$y'' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 2.$$

(a) Find the solution of this initial value problem.
(b) Plot y versus t and y' versus t on the same axes.
(c) Draw the phase portrait for the dynamical system equivalent to $y'' + 2y = 0$. Include the trajectory corresponding to the initial conditions $y(0) = 0$, $y'(0) = 2$.

30. The position of a certain spring-mass system satisfies the initial value problem

$$y'' + \tfrac{1}{4}y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 2.$$

(a) Find the solution of this initial value problem.
(b) Plot y versus t and y' versus t on the same axes.
(c) Draw the phase portrait for the dynamical system equivalent to $y'' + \frac{1}{4}y' + 2y = 0$. Include the trajectory corresponding to the initial conditions $y(0) = 0$, $y'(0) = 2$.

31. In the absence of damping, the motion of a spring-mass system satisfies the initial value problem

$$my'' + ky = 0, \quad y(0) = a, \quad y'(0) = b.$$

(a) Show that the kinetic energy initially imparted to the mass is $mb^2/2$ and that the potential energy initially stored in the spring is $ka^2/2$, so that initially the total energy in the system is $(ka^2 + mb^2)/2$.
(b) Solve the given initial value problem.
(c) Using the solution in part (b), determine the total energy in the system at any time t. Your result should confirm the principle of conservation of energy for this system.

32. If the restoring force of a nonlinear spring satisfies the relation

$$F_s(\Delta x) = -k\Delta x - \epsilon(\Delta x)^3,$$

where $k > 0$, then the differential equation for the displacement $x(t)$ of the mass from its equilibrium position satisfies the differential equation (see Problem 17, Section 4.1)

$$mx'' + \gamma x' + kx + \epsilon x^3 = 0.$$

Assume that the initial conditions are

$$x(0) = 0, \quad x'(0) = 1.$$

(a) Find $x(t)$ when $\epsilon = 0$ and also determine the amplitude and period of the motion.
(b) Let $\epsilon = 0.1$. Plot a numerical approximation to the solution. Does the motion appear to be periodic? Estimate the amplitude and period.
(c) Repeat part (c) for $\epsilon = 0.2$ and $\epsilon = 0.3$.
(d) Plot your estimated values of the amplitude A and the period T versus ϵ. Describe the way in which A and T, respectively, depend on ϵ.
(e) Repeat parts (c), (d), and (e) for negative values of ϵ.

4.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

We now return to the nonhomogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \tag{1}$$