# MAT267: HW3

Please do these problems and submit them by 11:59pm on Sunday (March 1).

*This document last updated Feb 26 to change the due date to March 1.*

1. The matrix

$$A = \begin{pmatrix} 2 & -1 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 2 & 1 \\ 0 & -1 & 2 & 2 \end{pmatrix}$$

has eigenvalues $2+i$, $2+i$, $2-i$, and $2-i$. It's not diagonalizable. It has eigenvalue-eigenvector pairs

$$\lambda_+ = 2+i, \qquad V_+ = \begin{pmatrix} 1 \\ -i \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \lambda_- = 2-i, \qquad V_- = \begin{pmatrix} 1 \\ i \\ 0 \\ 1 \end{pmatrix}$$

In the following you're going to first compute the Jordan canonical Form over the complex numbers. Then you'll compute the real canonical form over the real numbers.

(a) Find a vector $U_+$ so that

$$(A - \lambda_+ I)\, U_+ = V_+.$$

*Yes, you're going to have to do a tedious Gaussian elimination on a $4 \times 5$ complex matrix. At least it has nice entries.*

(b) Find a vector $U_-$ so that

$$(A - \lambda_- I)\, U_- = V_-.$$

*Please do this by being clever and don't do more Gaussian elimination!*

(c) Using the vectors $V_\pm$ and $U_\pm$ construct a complex matrix $P$ so that $P$ is invertible and

$$AP = P \begin{pmatrix} \lambda_+ & 1 & 0 & 0 \\ 0 & \lambda_+ & 0 & 0 \\ 0 & 0 & \lambda_- & 1 \\ 0 & 0 & 0 & \lambda_- \end{pmatrix}$$

(d) Write $V_+ = V_1 + iV_2$. Find $AV_1$ and $AV_2$ in terms of $\mathrm{Re}(\lambda_+)$, $\mathrm{Im}(\lambda_+)$, $V_1$, and $V_2$.

(e) Write $U_+ = V_3 + iV_4$. Find $AV_3$ and $AV_4$ in terms of $\mathrm{Re}(\lambda_+)$, $\mathrm{Im}(\lambda_+)$, $V_1$, $V_2$, $V_3$, and $V_4$.

(f) Using the vectors $V_1$, $V_2$, $V_3$, and $V_4$, construct a real matrix $P$ so that $P$ is invertible and

$$AP = P \begin{pmatrix} 2 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

(g) As a sanity check you should figure out how things would have changed had you done the real canonical form based on $V_- = V_1 - iV_2$ and $U_- = V_3 - iV_4$.

2. Consider the linear system

$$X' = AX = \begin{pmatrix} 3 & 0 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} X.$$

The matrix has eigenvalues 2, 2, 2, and 2. The time-$t$ map $\phi_t^A(X)$ is a linear:

$$\phi_t^A(X) = \mathcal{A}(t)\, X.$$

Another way of viewing this is: there's a time-dependent matrix $\mathcal{A}(t)$ so that the solution of the initial value problem $X' = AX$ with $X(0) = X_0$ can be written as

$$X(t) = \mathcal{A}(t)\, X_0.$$

Find the matrix $\mathcal{A}(t)$.

3. The $n \times n$ matrices $A$ and $B$ are similar if there is an invertible matrix $P$ so that

$$A = P\, B\, P^{-1}.$$

Assume $A$ and $B$ are similar matrices. Find a homeomorphism $H : \mathbb{R}^n \to \mathbb{R}^n$ so that $X' = AX$ and $X' = BX$ are conjugate linear systems.

4. Chapter 5, problem 9.

5. Chapter 5, problems 14 and 15. Note: you're not being asked for proofs, just give a brief reason for each item in problems 14 and 15.

6. Chapter 6, problem 5.

7. Chapter 6, problem 7.

8. Chapter 6, problems 12a, 12e, 12f, and 12j.

9. Chapter 6, problem 13.

10. Chapter 6, problem 14.