MAT267: HW4

Please do these problems and submit them by 11:59pm on Saturday (March 14).

This document last updated March 13 to clarify problem to replace a $x(t)$ with $X(t)$ in problem 5.

1. We know that if $G : (a, b) \to \mathbb{R}^n$ is a continuous function and $t_0 \in (a, b)$ then the initial value problem

$$\begin{cases} X' = A\, X + G(t) \\ X(t_0) = X_0 \end{cases}$$

has solution $X\colon (a, b) \to \mathbb{R}^n$ where

$$X(t) = e^{(t-t_0)A} X_0 + \int_{t_0}^{t} e^{(t-s)A} G(s)\; ds$$

Consider a linear system with continuous, time-dependent coefficients: that is, $A : (a, b) \to L(\mathbb{R}^n)$ is a continuous function. If you can find linearly independent solutions $X^{(i)}\colon (a, b) \to \mathbb{R}^n$ of the n linear systems

$$\begin{cases} X' = A(t)\, X \\ X(t_0) = E_i \end{cases}$$

then you can construct a matrix $\Psi\colon (a, b) \to L(\mathbb{R}^n)$ by putting $X^{(i)}$ into the ith column of $\Psi(t)$. By construction, $\Psi(t_0) = I$ and the initial value problem

$$\begin{cases} X' = A(t)\, X + G(t) \\ X(t_0) = X_0 \end{cases} \tag{1}$$

has solution $X\colon (a, b) \to \mathbb{R}^n$ where

$$X(t) = \Psi(t) X_0 + \int_{t_0}^{t} \Psi(t)\Psi(s)^{-1} G(s)\; ds. \tag{2}$$

   (a) Is it necessary that $\Psi(t)$ satisfies $\Psi(t_0) = I$? That is, if $\Psi(t)$ were simply built out of linearly independent solutions of $X' = A(t)X$, how would you modify the formula (2) so that you'd still have a solution of the initial value problem (1)?

   (b) Consider the linear second-order equation

$$p(t)\, y'' + q(t)\, y' + r(t)\, y = g(t) \tag{3}$$

   where p, q, r, and g are all continuous on (a, b) and p is strictly positive. Assume the homogenous ODE, $p(t)\, y'' + q(t)\, y' + r(t)\, y = 0$, has two solutions y_1 and y_2 on (a, b) and that there's no constant c so that $y_1 - c\, y_2$ is zero on (a, b).

   Write (3) as a first-order, linear system. Construct the matrix $\Psi(t)$. Using your work from part (a), construct the general solution of (3). Now, write your general solution in the form $y(t) = y_c(t) + u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$.

(c) In Paul's Online Notes (and in the MAT244 textbook and in your WebWork assignment) you're told to seek a particular solution y_p of (3) by assuming y_p has the form

$$y_p(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$$

and are given u_1 and u_2 as a pair of integrals involving p, q, r, g, y_1, y_1', y_2, and y_2'. Does your answer in part (b) agree with these integrals? If not, how would you modify the given integrals so that they agree?

(d) In Paul's Online Notes (and in the MAT244 textbook) the given integrals are found by finding a pair of differential equations that u_1 and u_2 satisfy. You derived u_1 and u_2 using (2). What is the connection between your solution and the the two differential equations that u_1 and u_2 satisfy?

(e) How would your answer to (b) change if p were strictly negative on (a, b)?

(f) Can you think of an example of (3) where p isn't nonnegative on (a, b) and yet the solution you constructed in part (b) still works?

2. Assume the matrix $A(t)$ in problem 1 is a 2×2 matrix. Find an ODE that's satisfied by the determinant of $\Psi(t)$. Solve the ODE. Conclude that if $\Psi(t_0)$ is invertible for some t_0 then $\Psi(t)$ is invertible for all $t \in (a, b)$.

3. Consider the ODE $p(t)y'' + q(t)\,y' + r(t)\,y = 0$ and the matrix $\Psi(t)$ from Problem 1b. Find an ODE that's satisfied by the determinant of $\Psi(t)$. (Do this using only $p(t)y'' + q(t)\,y' + r(t)\,y = 0$; do not do it by writing this as an $\vec{X}' = A(t)\vec{X}$ system and invoking your answer to Problem 2.) Solve the ODE. Conclude that $\Psi(t)$ is invertible on (a, b).

4. (a) Let A be an $n \times n$ matrix with real entries. Show that as $s \to 0$

$$det(I + sA) = (1 + s\,A_{1,1}) \cdot \cdots \cdot (1 + s\,A_{n,n}) + \mathcal{O}(s^2) = 1 + s\,Tr(A) + \mathcal{O}(s^2)$$

hence

$$\frac{d}{ds} det(I + sA)\bigg|_{s=0} = Tr(A).$$

Hint: Prove this by induction. For the 2×2 case, use

$$\begin{aligned} det(A + B) &= det(A_1 + B_1 | A_2 + B_2) = det(A_1 | A_2 + B_2) + det(B_1 | A_2 + B_2) \\ &= det(A_1 | A_2) + det(A_1 | B_2) + det(B_1 | A_2) + det(B_1 | B_2) \end{aligned}$$

where A_i denotes the ith column of A and B_i denotes the ith column of B.

(b) Let $B(s)$ be a smooth, matrix-valued function of s with $B(0) = I$. Use the previous exercise to show that

$$\frac{d}{ds} det(B(s))\bigg|_{s=0} = Tr(B'(0)).$$

(c) Let $C(t)$ be a smooth matrix-valued function and assume $C(0)$ is invertible. Use $B(s) = C(0)^{-1}\,C(s)$ to conclude

$$\frac{d}{ds} det(C(s))\bigg|_{s=0} = det(C(0))\,Tr(\,C(0)^{-1}\,C'(0)\,).$$

Generalize this away from derivatives evaluated at 0; do this by writing $C(s) = M(t+s)$ and use the above. If $M(t)$ is a smooth matrix-valued function and $M(t)$ is invertible, what is the right-hand-side of the ODE

$$\frac{d}{dt} det(M(t)) = ???$$

(d) Take $\Psi(t)$ from Problem 3 as the smooth matrix-valued function $C(t)$ above. Define the Wronskian $W(t) = det(\Psi(t))$ Prove that

$$W'(t) = Tr(A(t))\,W(t).$$

You may need to recall that if two matrices are similar then not only do they have the same determinant and eigenvalues but they also have the same trace.

5. Prove the following

Theorem 1. *Consider the initial value problem*

$$X' = F(X), \quad X(t_0) = X_0$$

where $X_0 \in \mathbb{R}^n$. Assume $F : \mathcal{O} \to \mathbb{R}^n$ is locally Lipschitz on the open set $\mathcal{O} \subset \mathbb{R}^n$. Let $X : (a, b) \to \mathbb{R}^n$ be the solution of the initial value problem. Assume there exists a closed and bounded set, $K \subset \mathcal{O}$ such that $X(t) \in K$ for all $t \in (a, b)$. Prove there exists $a_1 < a$ and $b_1 > b$ so that $x\colon (a_1, b_1) \to \mathbb{R}^n$ solves the initial value problem.

6. Consider the initial value problem

$$X' = \begin{pmatrix} x_2 \\ -x_1^3 \end{pmatrix}, \qquad X(t_0) = X_0$$

Let $X : (a, b) \to \mathbb{R}^n$ be the solution of the initial value problem. Demonstrate that your solution satisfies

$$\frac{d}{dt}\left(\frac{x_1(t)^4}{4} + \frac{x_2(t)^2}{2}\right) = 0.$$

Prove that the solution of the IVP exists for all time.

7. Consider the initial value problem

$$x' = x^{\beta}, \quad x(0) = 1$$

where $\beta \in (0, \infty)$. Let $x\colon (a, b) \to \mathbb{R}$ be the solution of the initial value problem. Show that the IVP has a solution on (a, ∞) if and only if $\beta \leq 1$.

8. Let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be C^1 and consider the IVP

$$X' = F(X), \quad X(t_0) = X_0.$$

Assume $X(t)$ is the solution of the IVP on (a, b)

(a) Show that for $t \in (a, b)$

$$\frac{d}{dt}\|X(t)\|^2 = 2\,X(t)\cdot F(X(t))$$

(b) Show that, if $\alpha > 0$ and $X(t) \neq 0$

$$\frac{d}{dt}\|X(t)\|^\alpha = \alpha\,\|X(t)\|^{\alpha-2}\,X(t)\cdot F(X(t))$$

(c) Assume there exists some $C < \infty$ and some $\beta < 1$ so that F satisfies the estimate

$$\|F(X)\| \leq C\,(1 + \|X\|)^\beta, \quad \forall X \in \mathbb{R}^n.$$

Show that there exists some $\alpha > 0$ and some $K < \infty$ such that if $\|X(t)\| \geq 1$ for all $t \in (a, b)$ then

$$\frac{d}{dt}\|X(t)\|^\alpha \leq K, \quad \forall t \in (a, b).$$

Use this to establish that the solution exists for all $t \in \mathbb{R}$.

9. Chapter 17, problem 1 *Make sure to try this problem even though only one of the cases is nice enough that you can find the general form for the Picard iterates. Certainly you can do the first few iterates and compare them to the exact solution (when you can find it). You don't have to hand it in.*

10. Chapter 17, problem 2 *Make sure you can do this problem; you don't have to hand it in.*

11. Chapter 17, problem 3 *Make sure you can do this problem; you don't have to hand it in.*

12. Chapter 17, problem 4 *Make sure you can do this problem; you don't have to hand it in.*

13. Chapter 17, problem 5 *Make sure you can do this problem; you don't have to hand it in.*