Sketch of HW4 Solutions

HW4 Solutions

1. a) It would change $\Psi(t)$ by $\Psi(t)P$ for some $P \in GL_n(\mathbb{R})$. Let $\Psi_1(t) = \Psi(t)P$, then

$$X(t) = \Psi_1(t)P^{-1}X_0 + \int_{t_0}^{t} \Psi_1(t)\Psi_1(s)^{-1}G(s)\,ds$$

b) Put $X = \begin{pmatrix} y \\ y' \end{pmatrix}$, now $X' = \begin{pmatrix} 0 & 1 \\ -\frac{r}{p} & -\frac{q}{p} \end{pmatrix} X + \underbrace{\begin{pmatrix} 0 \\ g(t) \end{pmatrix}}_{G(t)}$

We can solve the homogeneous $p(t)y'' + q(t)y' + r(t)y = g(t)$ and get solutions y_1, y_2 (linearly indep.)

So $\boxed{\Psi(t) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}}$ $\det \Psi(t) \neq 0$ as y_1, y_2 are linear indep.

Now we find $X(t) = \Psi(t)X_0 + \int_{t_0}^{t} \Psi(t)\Psi(s)^{-1}G(s)\,ds$, so the first component of this is

with initial value $X(t_0) = \boxed{\Psi(t_0)}X_0$

invertible matrix.

the solution y_c.

actually, it's

$$\Psi(t) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}^{-1}$$

e)

c) The differences between Paul's online notes and the formulae here are:

- online notes are for
$$y'' + q(t)y' + r(t)y = g(t)$$
So need to replace the g in the integrand from the online notes with $\frac{g}{p}$

- The online notes have to do with the general solution and involve indefinite integrals. The expressions here have to do with an initial value problem and involve definite integrals.

1d. Where did the 2 ODEs
$u_1'y_1 + u_2'y_2 = 0$ and
$u_1'y_1' + u_2'y_2' = \frac{g}{p}$ come from?

The particular solution is

$$X_p(t) = \Psi(t)\int_{t_0}^{t} \Psi(s)^{-1} G(s)\,ds$$
$$= \Psi(t)\int_{t_0}^{t} \frac{1}{W(s)}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{g(s)}{p(s)} \end{pmatrix} ds$$
$$= \Psi(t)\int_{t_0}^{t} \frac{1}{W(s)}\begin{pmatrix} \frac{-y_2 g}{p} \\ \frac{y_1 g}{p} \end{pmatrix} ds$$
$$= \Psi(t)\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \text{ where } u_1(t) = \int_{t_0}^{t} -\frac{y_2(s)g(s)}{W(s)\,p(s)}\,ds$$
$$u_2(t) = \int_{t_0}^{t} \frac{y_1(s)g(s)}{W(s)\,p(s)}\,ds$$

$$\Rightarrow X_p(t) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} y_1u_1 + y_2u_2 \\ y_1'u_1 + y_2'u_2 \end{pmatrix}$$

We know $X_p(t)$ is a particular solution and so

$$X_p' = \begin{pmatrix} 0 & 1 \\ -\frac{r}{p} & -\frac{q}{p} \end{pmatrix} X_p + \begin{pmatrix} 0 \\ g/p \end{pmatrix}$$

first component of this system is

$$(y_1u_1 + y_2u_2)' = y_1'u_1 + y_2'u_2$$

But the product rule tells us

$$(y_1u_1+y_2u_2)' = y_1'u_1+y_1u_1'+y_2'u_2+y_2u_2'$$

So we need $y_1u_1'+y_2u_2'=0$

This is the ODE that Paul's online notes pulled out of thin air.

~~otherwise~~ the product rule will be false.

Second component of + system is

$$(y_1'u_1+y_2'u_2)' = -\frac{r}{p}(y_1u_1+y_2u_2) - \frac{q}{p}(y_1'u_1+y_2'u_2)+\frac{g}{p}$$

$$p(y_1'u_1+y_2'u_2)' = -r(y_1u_1+y_2u_2)-q(y_1'u_1+y_2'u_2)+g$$

$$p[y_1''u_1+y_1'u_1'+y_2''u_2+y_2'u_2'] = (-ry_1-qy_1')u_1+(-ry_2-qy_2')u_2+g$$

$$py_1''u_1+py_1'u_1'+py_2''u_2+py_2'u_2' = (-ry_1-qy_1')u_1+(-ry_2-qy_2')u_2+g$$

we know
$$py_1''=-ry_1-qy_1'$$
$$py_2''=-ry_2-qy_2'$$

So $\boxed{py_1'u_1'+py_2'u_2'=g}$ needs to be true if x_p

is going to be a solution of

$$X_p' = \begin{pmatrix} 0 & 1 \\ -\frac{r}{p} & -\frac{q}{p} \end{pmatrix} X_p + G$$

1e If $p(t)$ were strictly negative there would be no change to the formulae because the important thing is that we can divide by $p(t)$.

1f $$p(t)y'' + q(t)y' + r(t)y = g(t)$$

take something like

$$p(t) = t$$
$$q(t) = te^t \qquad \text{on } \mathbb{R}$$
$$r(t) = t^2\sin(t)$$
$$g(t) = t^3$$

then you can still divide the $p(t)$ out and get

$$y'' + e^t y' + t\sin(t)y = t^2$$

No problemo! Finding a pair of

linearly independent solutions of

$$y'' + e^t y' + t\sin(t)\, y = 0$$

now that's another question...

#2 Assume the columns of $\Psi(t)$ are solutions of

$$X' = A(t)X.$$

What ODE does $\det(\Psi(t))$ satisfy?

$$\Psi(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \quad \det(\Psi(t)) = a(t)d(t) - b(t)c(t)$$

$$\begin{aligned}(\det(\Psi(t)))' &= a'd + ad' - b'c - bc' \\ &= a'd - bc' + ad' - b'c \\ &= \det\begin{pmatrix} a' & b \\ c' & d \end{pmatrix} + \det\begin{pmatrix} a & b' \\ c & d' \end{pmatrix}\end{aligned}$$

we know

$$\begin{pmatrix} a \\ c \end{pmatrix}' = (A_1 | A_2)\begin{pmatrix} a \\ c \end{pmatrix} = aA_1 + cA_2$$

$$\begin{pmatrix} b \\ d \end{pmatrix}' = (A_1 | A_2)\begin{pmatrix} b \\ d \end{pmatrix} = bA_1 + dA_2$$

and so..

$$\det\begin{pmatrix} a' & b \\ c' & d \end{pmatrix} = \det\left(aA_1 + cA_2 \,\middle|\, \begin{matrix} b \\ d \end{matrix}\right)$$
$$= a\det\left(A_1 \,\middle|\, \begin{matrix} b \\ d \end{matrix}\right) + c\det\left(A_2 \,\middle|\, \begin{matrix} b \\ d \end{matrix}\right)$$

$$\det\begin{pmatrix} a & b' \\ c & d' \end{pmatrix} = \det\left(\begin{matrix} a \\ c \end{matrix} \,\middle|\, bA_1 + dA_2\right)$$
$$= b\det\left(\begin{matrix} a \\ c \end{matrix} \,\middle|\, A_1\right) + d\det\left(\begin{matrix} a \\ c \end{matrix} \,\middle|\, A_2\right)$$

So
$$\left(\det(\Psi(t))\right)' = a\det\left(A_1 \,\middle|\, \begin{matrix} b \\ d \end{matrix}\right) - c\det\left(\begin{matrix} b \\ d \end{matrix} \,\middle|\, A_2\right)$$
$$- b\det\left(A_1 \,\middle|\, \begin{matrix} a \\ c \end{matrix}\right) + d\det\left(\begin{matrix} a \\ c \end{matrix} \,\middle|\, A_2\right)$$

$$= \det\left(A_1 \,\middle|\, a\begin{pmatrix} b \\ d \end{pmatrix} - b\begin{pmatrix} a \\ c \end{pmatrix}\right)$$
$$+ \det\left(-c\begin{pmatrix} b \\ d \end{pmatrix} + d\begin{pmatrix} a \\ c \end{pmatrix} \,\middle|\, A_2\right)$$

$$= \det\left(A_1 \,\middle|\, \begin{matrix} 0 \\ ad-bc \end{matrix}\right)$$
$$+ \det\left(\begin{matrix} ad-bc \\ 0 \end{matrix} \,\middle|\, A_2\right)$$

$$= (ad-bc)a_{11} + (ad-bc)a_{22}$$
$$= \det(\Psi(t))\,\mathrm{Tr}(A(t))$$

3. We have $W = \det \Psi(t) = y_1 y_2' - y_2 y_1'$

so $W'(t) = y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1'' =$

$$y_1\left(\frac{-r y_2 - q y_2'}{p}\right) - y_2\left(\frac{-r y_1 - q y_1'}{p}\right) =$$

$$\cancel{\frac{-r}{p} y_1 y_2} - q y_1 y_2'/p + \cancel{\frac{r y_1 y_2}{p}} + \frac{q y_1' y_2}{p} = \left(-\frac{q}{p}\right)\left(y_1 y_2' - y_1' y_2\right) = \left(\frac{-q}{p}\right) W(t).$$

so we have $W' = \left(\frac{-q}{p}\right) W$

Let $Y = W e^{\int \frac{q}{p}} \Rightarrow Y' = W' e^{\int \frac{q}{p}} + W\left(\frac{q}{p}\right) e^{\int \frac{q}{p}} = \left(W' + W\frac{q}{p}\right) e^{\int \frac{q}{p}} = 0$

so $Y = \text{Const.} \Rightarrow W = C e^{-\int \frac{q}{p}}$, now we know that y_1, y_2 are linearly independent

~~and so at $t = t_0$~~, ~~$\Psi(t)$ is invertible~~. Now if $C \neq 0$, W is always nonzero, but if $C = 0$, $W = 0$ on all (a,b) which means that y_1, y_2 weren't linearly indep., which is a contradiction.

4- a) $\det(I+sA) = \det(\delta_{ij} + s a_{ij}) \underset{\text{using permutations}}{=} 1 + s\,\mathrm{tr}A + (\mathcal{O}(s^2))$

we cannot choose (to get s. element) $s a_{ij}$ for one component of $i \neq j$ the permuted diagonal as we have to choose another one of this form so to get s^2.

so $\frac{d}{ds}\det(I+sA)\Big|_{s=0} = \mathrm{tr}\,A$.

b) Let $B(s) = \left(B_1(s) \,\middle|\, B_2(s) \,\middle|\, \cdots \,\middle|\, B_n(s)\right)$ then

$$(*) \quad (\det B(s))' = \underbrace{\det\left(B_1'(s) \,\middle|\, B_2(s) \,\middle|\, \cdots \,\middle|\, B_n(s)\right)}_{b_1(s)} + \underbrace{\det\left(B_1(s) \,\middle|\, B_2'(s) \,\middle|\, \cdots \,\middle|\, B_n(s)\right)}_{b_2(s)} + \cdots + \underbrace{\det\left(B_1(s) \,\middle|\, B_2(s) \,\middle|\, \cdots \,\middle|\, B_n'(s)\right)}_{b_n(s)}$$

but we know $B(0) = I$, so $b_1(0) = b_{11}'(0)$, $b_2(0) = b_{22}'(0)$, ...

so $(\det B(s))'\Big|_{s=0} = b_{11}'(0) + b_{22}'(0) + \cdots + b_{nn}'(0) = \mathrm{tr}(B'(0))$.

c) $(\det C(0)^{-1}C(s))'\Big|_{s=0} = \mathrm{tr}\left((C^{-1}(0)\,C(s))'\right)\Big|_{s=0} = \mathrm{tr}\left(C^{-1}(0)\,C'(0)\right)$

$\parallel$

$(\det C(0))^{-1}(\det C(s))'\Big|_{s=0} = \mathrm{tr}(C^{-1}(0)\,C'(0)) \Rightarrow (\det C(s))'\Big|_{s=0} = \det C(0)\ \mathrm{tr}(C^{-1}(0)\,C'(0))$.

It is a simple chain rule to change it to $M(t+s)$.

The ODE is the same as $(*)$ above.

d) $\cancel{\Psi(t) = A(t)\Psi(t) \Rightarrow \det\Psi(t)}$ By $(*)$, $\left(\text{let } \Psi(t) = (\tilde{A} \,|\, \tilde{B})\right)$

$(\det \Psi(t))' = \det(\tilde{A}' \,|\, \tilde{B}) + \det(\tilde{A} \,|\, \tilde{B}') = \det(A\tilde{A} \,|\, \tilde{B}) + \det(\tilde{A} \,|\, A\tilde{B}) \overset{\text{easily}}{=}$

$\mathrm{tr}(A)\det(\tilde{A} \,|\, \tilde{B}) = \mathrm{tr}\,A\ \det\Psi$.

Scanned with CamScanner

#5
The proof is exactly that given in the book, pages 398 - 399. In the book it's (α, β) and a compact set C. In the problem it's (a, b) and a compact set K but that's the only difference.

#6 $\quad X' = \begin{pmatrix} x_2 \\ -x_1^3 \end{pmatrix} \quad X(t_0) = X_0$

$\Rightarrow x_1' = x_2$ and $x_2' = -x_1^3$

Let $X : (a, b) \to \mathbb{R}^2$ be a solution of the IVP. Then

$$\frac{d}{dt}\left(\frac{1}{4}x_1(t)^4 + \frac{1}{2}x_2(t)^2\right)$$

$$= x_1(t)^3 x_1'(t) + x_2(t) x_2'(t)$$

$$= x_1(t)^3 [x_2(t)] + x_2(t)[-x_1(t)^3]$$

$$= 0$$

$$\Rightarrow \frac{1}{4}x_1(t)^4 + \frac{1}{2}x_2(t)^2 = \frac{1}{4}x_1(0)^4 + \frac{1}{2}x_2(0)^2$$

$$\forall t \in (a, b)$$

this means
$X(t)$ lies inside a compact set K.

What's K? One option is the "kinda elliptical" set in $\mathbb{R}^2$

$$K = \left\{ (x,y) \;\middle|\; \frac{1}{4}x^4 + \frac{1}{2}y^2 = M \right\}$$

where $M = \frac{1}{4}x_1(0)^4 + \frac{1}{2}x_2(0)^2$

Another option would be

$$\frac{1}{4}x_1(t)^4 + \frac{1}{2}x_2(t)^2 = M$$

$$\Rightarrow x_1(t)^4 \le 4M$$

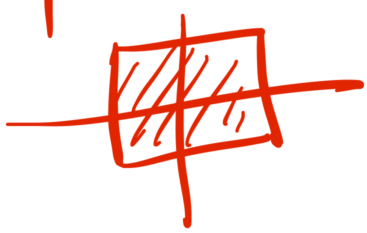
$$\Rightarrow |x_1(t)| \le \sqrt[4]{4M}$$

also $x_2(t)^2 \le 2M$

$$\Rightarrow |x_2(t)| \le \sqrt{2M}$$

So let

$$K = \left\{ (x,y) \;\middle|\; \max\{|x|,|y|\} \le \max\{\sqrt[4]{4M}, \sqrt{2M}\} \right\}$$

this is a  K

Because $x(t)$ will always stay within K, its maximal interval of existence (α, β) cannot be finite at either end, by the theorem on page 398 of the book.

#7. $x' = x^{\beta}$ $\quad x(0) = 1$

Assume $\beta > 0$.
Show the solution of the IVP has a solution on $(a, \infty) \iff \beta \leq 1$
(i.e. linear or sublinear growth

<u>case 1:</u> $\beta = 1$ then $x' = x$, $x(0) = 1$
has solution
$x(t) = e^{t}$
is defined on all $\mathbb{R}$.

<u>case 2:</u> $\beta \neq 1$. The ODE is separable.

$$\frac{dx}{dt}(t)\, x(t)^{-\beta} = 1$$

$$\frac{1}{1-\beta}\frac{d}{dt}x(t)^{1-\beta} = 1$$

$$\frac{d}{dt}x(t)^{1-\beta} = 1-\beta$$

$$x(t)^{1-\beta} = (1-\beta)t + x(0)^{1-\beta}$$

$$x(t)^{1-\beta} = (1-\beta)t + 1$$

$$x(t) = [1+(1-\beta)t]^{\frac{1}{1-\beta}}$$

- if $\beta < 1$ then $1-\beta > 0$ and as $t \uparrow \infty$, $(1+(1-\beta)t) \uparrow \infty$

  the exponent $\frac{1}{1-\beta}$ is positive and there's no problem with non-existence as t increases.

- if $\beta > 1$ then $1-\beta < 0$ and

  $1+(1-\beta)t = 0$ when $t = \frac{1}{\beta-1}$

  So as t increases to $\frac{1}{\beta-1}$

  $1+(1-\beta)t$ decreases to 0.

  The exponent $\frac{1}{1-\beta} < 0$ and so

  $x(t) \uparrow \infty$ as $t \uparrow \frac{1}{\beta-1}$. Solution cannot be extended.

#8 $\begin{cases} X' = F(X) \\ X(t_0) = X_0 \end{cases}$

Assume $X(t)$ is a solution on (a,b)

a) Show that for all $t \in (a,b)$

$$\frac{d}{dt}|X(t)|^2 = 2X(t) \cdot F(t).$$

ans: $|X(t)| = \sqrt{X(t) \cdot X(t)}$ so

$$\frac{d}{dt}|X(t)|^2 = \frac{d}{dt} X(t) \cdot X(t) = X'(t) \cdot X(t) + X(t) \cdot X'(t)$$

$= 2X(t) \cdot X'(t)$ (because $X(t) \in \mathbb{R}^n$ otherwise we would have $4\,\mathrm{Re}(X \cdot X') \ldots$)

$= 2X(t) \cdot F(X(t))$

$\forall t \in (a,b)$ because $X(t)$ is a solution of $X' = F(X)$.

b) Assume $\alpha > 0$ and $X(t) \neq \vec{0}$. Show that

$$\frac{d}{dt}|X(t)|^\alpha = \alpha |X(t)|^{\alpha-2} X(t) \cdot F(X(t))$$

$$\frac{d}{dt}\left(X(t) \cdot X(t)\right)^{\frac{\alpha}{2}} = \frac{\alpha}{2}\left(X(t) \cdot X(t)\right)^{\frac{\alpha}{2}-1} \frac{d}{dt}\left[X(t) \cdot X(t)\right]$$

$= \frac{\alpha}{2}|X(t)|^{\alpha-2}\left[2X(t) \cdot F(X(t))\right]$ by part a)

$= \alpha |X(t)|^{\alpha-2} X(t) \cdot F(X(t))$

c) Assume $\exists$ some $C<\infty$ and some $\beta<1$

$\ni \quad |F(x)| \le C(1+|x|)^{\beta} \quad \forall x \in \mathbb{R}^n$

Show that $\exists$ some $\alpha>0$ and $K<\infty$ so that if $|x(t)| \ge 1$ for all $t \in (a,b)$ then

$$\frac{d}{dt}|x(t)|^{\alpha} \le K \quad \forall t \in (a,b)$$

Use this to establish that the solutions $\exists$ for all time.

- Case 1: $\beta \le 0$ in which case

$$|F(x)| \le C(1+|x|)^{0} \Rightarrow |F(x)| \le C \quad (\text{if } \beta=0)$$

or

$$|F(x)| \le C(1+|x|)^{neg} \Rightarrow |F(x)| \le 2C$$
$$\forall |x| \ge 1$$

We seek $\alpha>0$ and $K<\infty$ so that

$$\frac{d}{dt}|x(t)|^{\alpha} \le K$$

We know $\frac{d}{dt}|x(t)|^{\alpha} = \alpha |x(t)|^{\alpha-2} x(t) \cdot F(x(t))$

so $\frac{d}{dt}|X(t)|^{\alpha} \le |\alpha| |X(t)|^{\alpha-2} |X(t)| |F(X(t))|$

$$= |\alpha| |X(t)|^{\alpha-1} |F(X(t))|$$

What do we know about $X(t)$ and $F(X(t))$?

- If $|X(t)| \le 1$ then we have $X(t)$ in a closed + bounded set, let
  $M := \max_{|X| \le 1} |F(X)|$.

  As long as $\alpha - 1 \ge 0$ we know that $|X(t)|^{\alpha-1} \le 1$ and so

  $$|X(t)| \le 1 \Rightarrow |\alpha| |X(t)|^{\alpha-1} |F(X(t))| \le |\alpha| M$$

- If $|X(t)| > 1$ then we have three cases. If $\beta = 0$ then we have $|F(X(t))| \le C$ and we can take

  $\alpha = 1 \Rightarrow \frac{d}{dt}|X(t)| \le 1 |X(t)|^{1-1} C = C$

If $\beta < 0$ then we have

$$|F(X)| \leq C(1+|X|)^{neg}$$

the magnitude of the vector field decays as $|X|$ as $|X| \to \infty$.

We know for any $\alpha \in \mathbb{R}$

$$\frac{d}{dt}|X(t)|^{\alpha} \leq |\alpha| \, |X(t)|^{\alpha-1} C(1+|X(t)|)^{\beta}$$

and by assumption $\beta < 0$. Choose

$\alpha - 1 = -\beta \Rightarrow \alpha - 1 > 0$ and so

$$|X(t)|^{\alpha-1} \leq (1+|X(t)|)^{\alpha-1}$$

$\therefore$ if $\beta < 0$ take $\alpha = 1-\beta$ and

$$\frac{d}{dt}|X(t)|^{1-\beta} \leq |1-\beta| (1+|X(t)|)^{-\beta} C (1+|X(t)|)^{\beta}$$
$$= |1-\beta| C$$

So we've got what we want:

if $\beta \leq 0$ then we take $\alpha = 1-\beta$

on $$\frac{d}{dt}|X(t)|^{1-\beta} \leq \max\{|1-\beta|M, |1-\beta|C\}$$

from control in $|X| \leq 1$ from control on $|X| \geq 1$

If $0<\beta<1$

In this case, we have

$$|F(x)| \le C(1+|x|)^{\beta}$$

the magnitude of the vector field grows with $|x|$. If we have chosen α so that $\alpha-1=-\beta$ then we get

$$\frac{d}{dt}|x(t)|^{1-\beta} \le |1-\beta|\,|x(t)|^{-\beta} C(1+|x|)^{\beta}$$

$$(\text{if } |x(t)|\ge 1)$$

$$= C|1-\beta| \frac{(1+|x(t)|)^{\beta}}{|x(t)|^{\beta}}$$

The function $g(y)=\dfrac{(1+y)^{\beta}}{y^{\beta}}$ is cts on $[1,\infty)$ and is bounded as $y\to\infty$. Therefore $\exists$ $\tilde{M}$ so that $g(y)\le\tilde{M}$ for $y\ge 0$.

Hence $\dfrac{d}{dt}|x(t)|^{1-\beta} \le C|1-\beta|\tilde{M}$ if $|x(t)|\ge 1$

Note: this argument works for any

$\beta > 0$. We need $\beta < 1$ so that we have $\alpha = 1 - \beta > 0$ which is what we needed to control $\frac{d}{dt}|x(t)|^{\alpha}$ if $|x(t)| < 1$.

Summing it all up, if $\beta < 1$ then

$$\frac{d}{dt}|x(t)|^{1-\beta} \leq \max\{\overbrace{|1-\beta| M}^{\text{from control when } |x| \leq 1},\ \underbrace{c|1-\beta|\tilde{M}}_{\text{from control when } |x| \geq 1}\}$$

for all $t \in (a, b)$ the maximal interval of existence

This proves that if $\exists\ \beta < 1$, $c < \infty$ so that

$$|F(x)| \leq C(1 + |x|)^{\beta} \quad \forall |x| \geq 1$$

then

$$\frac{d}{dt}|x(t)|^{1-\beta} \leq \hat{M} < \infty \quad \forall t \in (a, b)$$

in the maximal interval of existence.

The second part of the question was to show that the maximal interval of existence has no upper bound: it's (a, ∞).

Because

$$\frac{d}{dt} |x(t)|^{1-\beta} \le M < \infty,$$

we know

$$|x(t)|^{1-\beta} \le |x_0|^{1-\beta} + M(t - t_0)$$

the RHS is finite for every t and so if the maximal interval of existence (a, b) has $b < \infty$ then we have

$$|x(t)|^{1-\beta} \text{ bounded on } [t_0, b)$$

$$\beta < 1 \Rightarrow |x(t)| \text{ is bounded on } [t_0, b)$$

1. ~~Le~~ $\{x(t) \mid t \in [t_0, b)\} \subset C$

where C is a compact set.. in $\mathbb{R}^n$

Because F is assumed C^1 on all of $\mathbb{R}^n$, it's locally Lipschitz on all of $\mathbb{R}^n$ and we can apply the theorem from page 398 of the book. $\Rightarrow$ b cannot be finite and the solution's maximal interval of existena is (a, ∞), as desired. //

Note: if $F: \mathcal{O} \to \mathbb{R}^n$ then we would need to worry about $b < \infty$ of the possibility of $|x(t)|$ being bounded but $x(t)$ getting close to the boundary of the open set $\mathcal{O}$.