Solutions to 6th HW assignment

1] exercise 3 from page 184 of H+S+D.

Consider $x' = f(x; a)$
for which $f(x_0, a) = 0$ and $\frac{\partial f}{\partial x}(x_0, a) \neq 0$.
Prove that for ε sufficiently small, the
$$x' = f(x, a+\varepsilon)$$
has an equilibrium point $x_0(\varepsilon)$ where
$x_0(0) = x_0$ and $\varepsilon \to x_0(\varepsilon)$ is smooth.

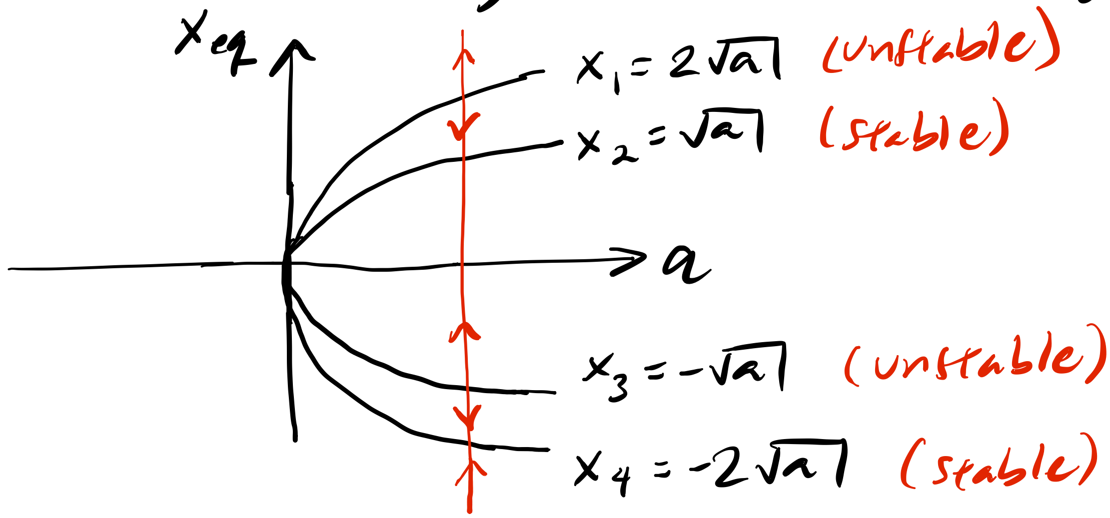
ANS: This is an application of the implicit function theorem:

IFT: Let $A \subseteq \mathbb{R} \times \mathbb{R}$ be an open set and let $f: A \to \mathbb{R}$ be C^p. Suppose $(x_0, a) \in A$ and $f(x_0, a) = 0$. If $\frac{\partial f}{\partial x}(x_0, a) \neq 0$ then $\exists$ an open interval U of x_0 and an open interval V containing a and there's a unique function $x: V \to U$ so that $f(x(a), a) = 0$ for all $a \in V$. Furthermore, x is of class C^p.

So the IFT gives us $x: (a-\varepsilon, a+\varepsilon) \to U$, where $x(a) = x_0$. To meet the book's notation, $x_0(\varepsilon) := x(a+\varepsilon)$.

#21 Ch 8 problem 6

Give an example of a family of differential equations $x' = f_a(x)$ for which there are no equilibrium solutions if $a<0$, a single equilibrium if $a=0$ and 4 equilibrium solutions if $a>0$. Sketch the bifurcation diagram for this family.



$$(x-x_1)(x-x_4)(x-x_2)(x-x_3)$$
$$= (x-2\sqrt{|a|})(x+2\sqrt{|a|})(x-\sqrt{|a|})(x+\sqrt{|a|})$$
$$= (x^2-4a)(x^2-a)$$

$$x' = f_a(x) = (x^2-4a)(x^2-a)$$

$a<0 \Rightarrow$ no equilibrium solutions

$a=0 \Rightarrow x=0$ only eq. solution

$a>0 \Rightarrow 4$ equilibrium solutions

#3

a) $x' = \sin(x)$ has equilibrium solutions
$x = n\pi, \quad n \in \mathbb{Z}$

b) $x_{ss} = \pi$ $\quad y(t) = x(t) - \pi$
$$y' = x' = \sin(x)$$
$$= \sin(y + \pi)$$
$$= -\sin(y)$$

c) $y' = -\sin(y)$
$$= -y + [y - \sin(y)] = -y + h(y)$$
$$= -\lambda y + h(y) \text{ where } \lambda = 1$$

claim: $\lim\limits_{y \to 0} \dfrac{h(y)}{y} = 0$

$$\lim_{y \to 0} \frac{y - \sin(y)}{y} = \lim_{y \to 0} \frac{1 - \cos(y)}{1} = 0$$
by l'Hôpital's rule.

d) $y' = -\lambda y + h(y) \Rightarrow yy' = -\lambda y^2 + y h(y)$
$$\Rightarrow \left(\tfrac{1}{2}y^2\right)' = y^2\left[-\lambda + \frac{h(y)}{y}\right] =: \Psi(y)$$

$\tilde{\Psi}(y) = -\frac{\lambda}{2} y^2$ show $\exists \delta > 0 \ni |y| < \delta$
$$\Rightarrow \Psi(y) < \tilde{\Psi}(y)$$

We have $\Psi(y) < \tilde{\Psi}(y) \iff y^2\left[-\lambda + \frac{h(y)}{y}\right] < -\frac{\lambda}{2}y^2$

$$\iff -\lambda + \frac{h(y)}{y} < -\frac{\lambda}{2}$$

$$\iff \frac{h(y)}{y} < \frac{\lambda}{2} = \frac{1}{2}$$

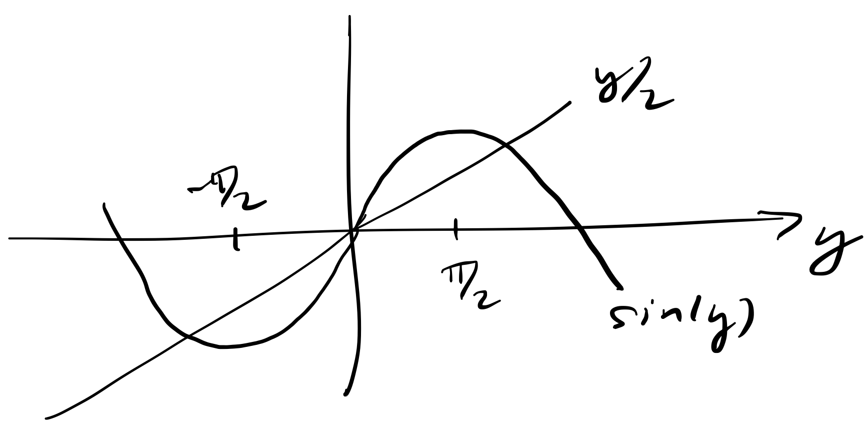
so we need to find $\delta > 0$ so that

$$|y| < \delta \Rightarrow \frac{y - \sin(y)}{y} < \frac{1}{2}$$

if $y > 0$ then need $\sin(y) > y/2$

if $y < 0$ then need $\sin(y) < y/2$

So $|y| < \pi/2 \Rightarrow \frac{y - \sin(y)}{y} < \frac{1}{2}$ for sure



e) Assume $y_0 \in (-\pi/2, \pi/2)$. Assume $y(t)$ solves the IVP $y' = -\sin(y)$, $y(0) = y_0$ let (α, β) be the maximal interval of existence of $y(t)$.

$y(t)$ is continuous so $\exists$ $(c,d) \ni 0$ so that $y(t) \in (-\pi/2, \pi/2)$ $\forall\, t \in (c,d)$.

$\Rightarrow$ For all $t \in (c,d)$

$$\left(\tfrac{1}{2}y(t)^2\right)' = \Psi(y(t)) < \tilde{\Psi}(y(t)) = -\lambda\left(\tfrac{1}{2}y(t)\right)^2$$

$$\Rightarrow \left(y(t)^2\right)' + \lambda\left(y(t)\right)^2 < 0$$

$$\Rightarrow \quad \left(e^{\lambda t}y(t)^2\right)' < 0$$

$$\Rightarrow \quad e^{\lambda t}y(t)^2 \text{ is a decreasing}$$

function

$$e^{\lambda t}y(t)^2 < y_0^2 \quad \text{for } t \in (0,d)$$

$$\Rightarrow \quad 0 \leq y(t)^2 < y_0^2 e^{-\lambda t} \quad \text{for } t \in (0,d)$$

This means for that $\forall\, t \in [0,d)$ $y(t) \in [-\pi/2, \pi/2]$ which is a compact set. $\Rightarrow d = \beta$ and $\beta = \infty$.

By squeeze theorem, $y(t)^2 \to 0$ as $t \to \infty$ therefore $y(t) \to 0$ as $t \to \infty$

#4

$$X' = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix} X + H(X)$$

where $\alpha > 0$ and $|H(X)|/|X| \to 0$ as $|X| \to 0$

If $X(t)$ is a solution then

$$X(t) \cdot X'(t) = X(t) \cdot \begin{pmatrix} -\alpha X_1(t) + \beta X_2(t) \\ -\beta X_1(t) - \alpha X_2(t) \end{pmatrix} + X(t) \cdot H(X(t))$$

$$\left(\tfrac{1}{2}|X(t)|^2\right)' = \begin{matrix} -\alpha X_1(t)^2 + \beta X_1(t) X_2(t) \\ -\beta X_1(t) X_2(t) - \alpha X_2(t)^2 \end{matrix} + X(t) \cdot H(X(t))$$

$$\left(\tfrac{1}{2}|X(t)|^2\right)' = -\alpha |X(t)|^2 + X(t) \cdot H(X(t))$$

Now $$|X(t) \cdot H(X(t))| \le |X(t)|\,|H(X(t))|$$

$$= |X(t)|^2 \frac{|H(X(t))|}{|X(t)|}$$

$$\Rightarrow \left(\tfrac{1}{2}|X(t)|^2\right)' \le -\alpha |X(t)|^2 + |X(t)|^2 \frac{|H(X(t))|}{|X(t)|}$$

$$\Rightarrow \left(\frac{1}{2}|X(t)|^2\right)' \le |X(t)|^2\left\{-\alpha + \frac{|H(X(t))|}{|X(t)|}\right\}$$

because $\frac{|H(X)|}{|X|} \to 0$ as $|X| \to 0$ $\exists\, \delta > 0$

So that $|X| < \delta \Rightarrow \frac{|H(X)|}{|X|} < \frac{\alpha}{2}$

$\Rightarrow$ if $X(t) \in B_\delta(0)$ then

$$\left(\frac{1}{2}|X(t)|^2\right)' < |X(t)|^2\left\{-\alpha + \frac{\alpha}{2}\right\}$$

$$= -\frac{\alpha}{2}|X(t)|^2$$

By the same argument as for problem 3, if $X_0 \in B_\delta(0)$ then the solution exists for all $t > 0$ and

$$0 \le |X(t)|^2 < |X_0|^2 e^{-\alpha t}$$

hence $|X(t)| \to 0$ as $t \to \infty$ and therefore $X(t) \to \vec{0}$ as $t \to \infty$.

#5] Consider

$$X' = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} X + H(X)$$

where $\lambda > 0$ and $\frac{|H(X)|}{|X|} \to 0$ as $|X| \to 0$

as suggested on page 70 of the book, we're going to change coordinates.

Let $T = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$

and $Y(t) = T^{-1} X(t)$

$$\Rightarrow Y'(t) = T^{-1} X'(t)$$

$$= T^{-1} \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} X(t) + T^{-1} H(X(t))$$

$$= T^{-1} \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} T \cdot Y(t) + T^{-1} H(TY(t))$$

$$\Rightarrow Y'(t) = \begin{pmatrix} -\lambda & \varepsilon \\ 0 & -\lambda \end{pmatrix} Y(t) + T^{-1} H(TY(t))$$

We know that $|H(X)| / |X| \to 0$

as $|X| \to 0$. Now argue that

$$\frac{|T^{-1}H(TY)|}{|Y|} \to 0 \text{ as } |Y| \to 0.$$

i.e. a linear change of coordinates doesn't change the fact that $H(X)$ goes to zero faster than linearly.

We have to be careful now with our norms. In all the above $|\ |$ denotes the L^2 norm. Now let's be careful.

$$\|X\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|X\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

As described in the solutions to HW3,

$$\frac{1}{\sqrt{n}}\|X\|_1 \le \|X\|_2 \le \sqrt{n}\,\|X\|_1$$

and if C is a matrix with

$$\|C\|_\infty = \max_{1 \le i,j \le n} \{|C_{ij}|\}$$

then $\|CX\|_1 \le \|C\|_\infty \|X\|_1$

because in $\mathbb{R}^2$

So $$\frac{\|T^{-1}H(TY)\|_2}{\|Y\|_2} \le \frac{\sqrt{2}\,\|T^{-1}H(TY)\|_1}{\|Y\|_2}$$

$$\le \sqrt{2}\,\frac{\|T^{-1}\|_\infty \|H(TY)\|_1}{\|Y\|_2}$$

$$\le \frac{\sqrt{2}\,\|T^{-1}\|_\infty \left[\sqrt{2}\,\|H(TY)\|_2\right]}{\|Y\|_2}$$

$$= 2\|T^{-1}\|_\infty \frac{\|H(TY)\|_2}{\|TY\|_2} \, \frac{\|TY\|_2}{\|Y\|_2}$$

These terms are controlled by some upper bound

This part, we like because we know that it goes to zero as TY goes to $\vec{0}$.

$$
\begin{aligned}
\|TY\|_2 &\le \sqrt{2}\,\|TY\|_1 \\
&\le \sqrt{2}\,\|T\|_\infty \|Y\|_1 \\
&\le \sqrt{2}\,\|T\|_\infty \left(\sqrt{2}\,\|Y\|_2\right)
\end{aligned}
$$

$$
\Rightarrow \quad \frac{\|TY\|_2}{\|Y\|_2} \le 2\|T\|_\infty
$$

So we have

$$
\frac{\|T^{-1}H(TY)\|_2}{\|Y\|_2} \le 4\|T\|_\infty \|T^{-1}\|_\infty \frac{\|H(TY)\|_2}{\|TY\|_2}
$$

and because we know

$$
\frac{\|H(Z)\|_2}{\|Z\|_2} \to 0 \quad \text{as} \quad Z \to \vec{0}
$$

we therefore know that

$$
\frac{\|T^{-1}H(TY)\|_2}{\|Y\|_2} \to 0 \quad \text{as} \quad Y \to \vec{0}.
$$

So much work! ☺ But now we've changed the challenging system

$$X' = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} X + H(X)$$

into an easier system

$$Y' = \begin{pmatrix} -\lambda & \varepsilon \\ 0 & -\lambda \end{pmatrix} Y + T^{-1} H(TY)$$

$$Y(t) \cdot Y'(t) = Y(t) \cdot \begin{bmatrix} -\lambda y_1(t) + \varepsilon y_2(t) \\ -\lambda y_2(t) \end{bmatrix}$$
$$+ Y(t) \cdot \left(T^{-1} H(TY(t))\right)$$

$$\left(\tfrac{1}{2}|Y(t)|^2\right)' = -\lambda y_1(t)^2 + \varepsilon y_1(t) y_2(t)$$
$$- \lambda y_2(t)^2$$
$$+ Y(t) \cdot \left(T^{-1} H(TY(t))\right)$$

$$= -\lambda |Y(t)|^2 + \varepsilon y_1(t) y_2(t)$$
$$+ Y(t) \cdot \left(T^{-1} H(TY(t))\right)$$

we know how to handle the $Y \cdot (\text{nonlinear})$ term but what about the $\varepsilon y_1(t) y_2(t)$ term?

<u>Idea</u>: if we choose ε small enough then we can "control" this term using the $-\lambda |Y(t)|^2$ in the same spirit as using that term to control the $Y(t) \cdot (\text{nonlinear})$ part.

$$\varepsilon y_1(t) y_2(t) \leq |\varepsilon| |y_1(t)| |y_2(t)|$$

$$\leq |\varepsilon| \sqrt{y_1^2 + y_2^2} \sqrt{y_1^2 + y_2^2}$$

$$= |\varepsilon| |Y(t)|^2$$

$$\Rightarrow \left(\tfrac{1}{2}|Y(t)|^2\right)' \le -\lambda |Y(t)|^2 + |\Sigma|\,|Y(t)|^2 + Y(t)\cdot\left(T^{-1}H(TY(t))\right)$$

$$\le -\lambda|Y(t)|^2 + |\Sigma|\,|Y(t)|^2 + |Y(t)|\,|T^{-1}H(TY)|$$

$$= -\lambda|Y(t)|^2 + |\Sigma|\,|Y(t)|^2 + |Y(t)|^2 \frac{|T^{-1}H(TY(t))|}{|Y(t)|}$$

$$\le -\lambda|Y(t)|^2 + |\Sigma|\,|Y(t)|^2 + |Y(t)|^2\left(4\|T\|_\infty \|T^{-1}\|_\infty \frac{\|H(TY)\|_2}{\|TY\|_2}\right)$$

$$= |Y(t)|^2\left\{-\lambda + |\Sigma| + 4\|T\|_\infty\|T^{-1}\|_\infty \frac{\|H(TY)\|_2}{\|TY\|_2}\right\}$$

Almost done!

(1) Choose $\varepsilon = \frac{\lambda}{4}$

(2) Choose $\delta > 0$ so that

$$\|TY\| < \delta \Rightarrow$$

$$4\|T\|_\infty \|T^{-1}\|_\infty \frac{\|H(TY)\|_2}{\|TY\|_2} < \frac{\lambda}{4}.$$

With this choice of ε and δ we have

$$\|TY\| < \delta \Rightarrow \left(\frac{1}{2}|Y(t)|^2\right)' \leq -\frac{\lambda}{2}|Y(t)|^2$$

It then follows that if Y_0 is such that $TY_0 \in B_\delta(0)$

then $|Y(t)|^2 \leq |Y_0|^2 e^{-\lambda t}$ and so solutions exist for all $t \geq 0$ and $Y(t) \to 0$.

Now to transform back to the original coordinates!

$$Y(t) = T^{-1}X(t) \iff TY(t) = X(t).$$

So we have

$$X_0 \in B_\delta(\vec{0}) \Rightarrow \|T^{-1}X(t)\|^2 \leq \|T^{-1}X_0\|^2 e^{-\lambda t}$$

for all $t > 0$ hence $X(t) \to \vec{0}$

If we want finer information we can use $T = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \Rightarrow T^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon \end{pmatrix}$

$$\Rightarrow \quad T^{-1}X(t) = \begin{pmatrix} x_1(t) \\ \frac{1}{\varepsilon}x_2(t) \end{pmatrix}$$

and $0 \leq \left(x_1(t)^2 + \frac{1}{\varepsilon^2}x_2(t)^2\right) \leq \left(x_{10}^2 + \frac{1}{\varepsilon^2}x_{20}^2\right)e^{-\lambda t}$

#6/ §8, problem 11.

$X' = F(X)$ with $F(X^*) = \vec{0}$.

Show $\exists$ a change of coordinates that converts this to

$$Y' = AY + G(Y)$$

where A is the JCF of $DF(X^*)$ and $\frac{|G(Y)|}{|Y|} \to 0$ as $|Y| \to 0$

We know

$$DF(X^*) = PAP^{-1} \text{ where } A \text{ is in JCF.}$$

Let

$$Y = P^{-1}(X - X^*)$$

Then

$$X' = F(X) = DF(X^*)(X - X^*) + H(X)$$

$$\text{where } \frac{|H(X)|}{|X - X^*|} \to 0 \text{ as } X \to X^*$$

$$\Rightarrow X' = PAP^{-1}(X - X^*) + PP^{-1}H(X)$$

$$\Rightarrow P^{-1}X' = AP^{-1}(X - X^*) + P^{-1}H(X)$$

$$\Rightarrow Y' = AY + \underbrace{P^{-1}H(PY + X^*)}_{\text{this is } G(Y)}$$

does $\frac{|G(Y)|}{|Y|} \to 0$ as $\vec{Y} \to \vec{0}$?

does $\frac{|P^{-1}H(PY + X^*)|}{|Y|} \to 0$ as $Y \to 0$?

$PY + X^* = X$ and $Y = P^{-1}(X - X^*)$

does $\dfrac{|P^{-1}H(X)|}{|P^{-1}(X-X^*)|} \to 0$ as $X \to X^*$?

we know

$$\|P^{-1}H(X)\|_2 \leq 2\|P^{-1}\|_\infty \|H(X)\|_2$$

and we know

$$\|X-X^*\|_2 = \|PP^{-1}(X-X^*)\|_2 \leq 2\|P\|_\infty \|P^{-1}(X-X^*)\|_2$$

$$\Rightarrow \frac{1}{\|P^{-1}(X-X^*)\|_2} \leq \frac{2\|P\|_\infty}{\|X-X^*\|_2}$$

Therefore

$$\frac{|P^{-1}H(X)|}{|P^{-1}(X-X^*)|} \leq 4\|P^{-1}\|_\infty \|P\|_\infty \frac{|H(X)|}{|X-X^*|}$$

And so $\dfrac{|G(Y)|}{|Y|} \to 0$ as $Y \to 0$

because we know $\dfrac{|H(X)|}{|X-X^*|} \to 0$ as $X \to X^*$.

#7

a) $(X,Y)_B := \langle BX, BY \rangle$ where $\langle \cdot,\cdot \rangle$ is the usual L^2 inner product.

$(\cdot,\cdot)_B$ is an inner product on $\mathbb{R}^n$ if

i) $(\alpha X, Y)_B = \alpha (X,Y)_B \quad \forall \alpha \in \mathbb{R}, \forall X, Y \in \mathbb{R}^n$

ii) $(X+Y, Z)_B = (X,Z)_B + (Y,Z)_B \quad \forall X, Y, Z \in \mathbb{R}^n$

iii) $(X,X)_B = 0 \iff X = \vec{0}$

i) $$\begin{aligned} (\alpha X, Y)_B &= \langle B(\alpha X), BY \rangle \\ &= \langle \alpha BX, BY \rangle \\ &= \alpha \langle BX, BY \rangle = \alpha (X,Y)_B \end{aligned}$$

ii) $$\begin{aligned} (X+Y, Z)_B &= \langle B(X+Y), BZ \rangle \\ &= \langle BX + BY, BZ \rangle \\ &= \langle BX, BZ \rangle + \langle BY, BZ \rangle \\ &= (X,Z)_B + (Y,Z)_B \end{aligned}$$

iii) $(X,X)_B = \langle BX, BX \rangle = 0$

$\iff BX = \vec{0}$

we want $BX = \vec{0} \iff X = \vec{0}$

i.e. we must have B is invertible.

b) Show $\exists\, a, b \in (0, \infty)$ $\ni$

$$a\|X\|_2 \leq \|X\|_B \leq b\|X\|_2.$$

From the work in previous problems,

$$\|BX\|_2 \leq n\,\|B\|_\infty\,\|X\|_2$$

Also $\|X\|_2 = \|B^{-1}BX\|_2 \leq n\|B^{-1}\|_\infty \|BX\|_2$

$$\Longrightarrow \frac{1}{n\|B^{-1}\|_\infty}\|X\|_2 \leq \|BX\|_2 \leq n\|B\|_\infty \|X\|_2$$

so take $a = \dfrac{1}{n\|B^{-1}\|_\infty}$, $b = n\|B\|_\infty$ and

use $\|X\|_B = \|BX\|_2$

c) Consider $X' = F(X)$ where $F(0) = 0$

$F(X) = DF(0)X + H(X)$.

Show that if F is C^2 in a nbd $\mathcal{O}$ of $\vec{0}$ and if $\delta > 0$ is such that $B_\delta(\vec{0}) \subset \mathcal{O}$ then $\exists\, C_\delta < \infty$ so that

$$\|H(X)\|_2 \leq C_\delta \|X\|_2^2$$

i.e. $\|H(X)\|_2$ goes to zero at least quadratically fast.

This comes from the Lagrange form of the remainder in Taylor's theorem:

[Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is C^2 on an open convex set S. If $\vec{a} \in S$ and $\vec{a} + \vec{h} \in S$ then

$$f(\vec{a} + \vec{h}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot \vec{h} + R_{a,2}(\vec{h})$$

where $R_{\vec{a},2}(\vec{h}) = \sum_{|\vec{\alpha}|=2} \partial^{\vec{\alpha}} f(\vec{a}+c\vec{h}) \frac{\vec{h}^{\vec{\alpha}}}{\vec{\alpha}!}$

for some $c \in (0,1)$

$\vec{\alpha} \in \mathbb{N}^n$ and $\vec{\alpha} = 2\vec{e}_j$ then $\partial^{\vec{\alpha}} f = \frac{\partial^2 f}{\partial x_j^2}$

$|\vec{\alpha}| = \sum_{i=1}^{n} \alpha_i$

and $\vec{h}^{\vec{\alpha}} = (h_j)^2$

$\vec{\alpha} = \vec{e}_i + \vec{e}_j$ then $\partial^{\vec{\alpha}} f = \frac{\partial^2 f}{\partial x_i \partial x_j}$

and $\vec{h}^{\vec{\alpha}} = h_i h_j$

So the remainder is controlled by the second derivatives evaluated at somewhere on the segment connecting $\vec{a}$ to $\vec{h}$.

In our case, $\vec{a} = \vec{0}$ and $\vec{h} = \vec{x}$ and we just bound the derivatives on $\overline{B_\delta(\vec{0})}$

The above is for $f: \mathbb{R}^n \to \mathbb{R}$. In our case we have $F: \mathbb{R}^n \to \mathbb{R}^n$ and so it's a matter of applying Taylor's theorem to each component of F. The remainder term will involve evaluating at some point $c_i\vec{x}$ where $c_i \in (0,1)$ $i=1 \dots n$. Possibly a different point for each component but it doesn't matter because we're doing a bound on all second derivatives on the ball $\overline{B_\delta(\vec{0})}$. $M_\delta = \max_{1\le i\le n} \max_{x\in \overline{B_\delta}} \max_{|\vec{\alpha}|=2} |\partial^{\vec{\alpha}} f_i(x)|$

For one component of $H(\vec{x})$ we have

$$h_i(\vec{x}) = \sum_{|\vec{\alpha}|=2} \partial^{\vec{\alpha}} f_i(c_i\vec{x}) \frac{\vec{x}^{\vec{\alpha}}}{\vec{\alpha}!}$$

$$|h_i(x)| \le \sum_{|\vec{\alpha}|=2} |\partial^{\vec{\alpha}} f_i(c_i\vec{x})| \frac{|\vec{x}^{\vec{\alpha}}|}{\vec{\alpha}!}$$

$$\le \sum_{|\vec{\alpha}|=2} M_\delta \frac{|\vec{x}^{\vec{\alpha}}|}{\vec{\alpha}!} \le \sum_{|\vec{\alpha}|=2} M_\delta |\vec{x}^{\vec{\alpha}}|$$

$$\le \sum_{|\vec{\alpha}|=2} M_\delta \|\vec{x}\|^{|\vec{\alpha}|} = \|\vec{x}\|^2 \sum_{|\vec{\alpha}|=2} M_\delta$$

$$= \|\vec{x}\|^2 M_\delta \frac{n(n+1)}{2}$$

So $\|H(X)\|_1 = \sum_{i=1}^{n} |h_i(X)| \leq \sum_{i=1}^{n} \|\vec{x}\|^2 M_\delta \frac{n(n+1)}{2}$

$$= M_\delta \frac{n^2(n+1)}{2} \|X\|_2^2$$

and $\|H(X)\|_2 \leq \sqrt{n} \|H(X)\|_1$

$$\Rightarrow \|H(X)\|_2 \leq C_\delta \|X\|_2^2$$

where $C_\delta = M_\delta \frac{\sqrt{n}\, n^2(n+1)}{2}$

d) Assume $DF(\vec{0})$ is diagonalizable and all of its eigenvalues are negative with $\lambda_n \leq \lambda_{n-1} \leq \dots \leq \lambda_1 < 0$

Find B so that

$$(DF(\vec{0})X, X)_B \leq \lambda_1 (X, X)_B$$

ans: $DF(\vec{0}) = PJP^{-1}$ where J is diagonal

take $B = P^{-1}$.

$$(DF(\vec{0})X, X)_{P^{-1}} = \langle P^{-1} DF(\vec{0})X, P^{-1}X \rangle$$
$$= \langle J P^{-1}X, P^{-1}X \rangle$$

in general, $\langle JY, Y \rangle = \sum (JY)_i Y_i$
$$= \sum \lambda_i Y_i Y_i$$
$$\leq \lambda_1 \sum Y_i^2$$
$$= \lambda_1 \langle Y, Y \rangle$$

So $(DF(\vec{0})X, X)_B \leq \lambda_1 \langle BX, BX \rangle$
$$= \lambda_1 (X, X)_B$$

as desired.

e) Take δ smaller if necessary and show that if $Z(t)$ is a solution of $X' = F(X)$ w/ $Z_0 \in B_\delta(\vec{0})$ then

$$\frac{d}{dt}(Z(t), Z(t))_B \leq \lambda_1 (Z(t), Z(t))_B$$

and conclude that

$$\|Z(t)\|_B^2 \leq \|Z_0\|_B^2 e^{\lambda_1 t}$$

hence $Z(t) \to \vec{0}$ as $t \to \infty$.

$$\frac{d}{dt}(Z(t), Z(t))_B = \frac{d}{dt}\langle BZ(t), BZ(t)\rangle$$

$$= 2\langle BZ'(t), BZ(t)\rangle$$

$$= 2(Z'(t), Z(t))_B$$

$$= 2\left(DF(\vec{0})Z(t) + H(Z(t)), Z(t)\right)_B$$

$$
\begin{aligned}
&= 2\left(DF(\vec{0})Z(t), Z(t) \right)_B \\
&\qquad + 2\left(H(Z(t)), Z(t) \right)_B \\
&\leq 2\lambda_1 \|Z(t)\|_B^2 \\
&\qquad + 2\left(H(Z(t)), Z(t) \right)_B \\
&= 2\lambda_1 \|Z(t)\|_B^2 \\
&\qquad + 2\langle BH(Z(t)), BZ(t) \rangle \\
&\leq 2\lambda_1 \|Z(t)\|_B^2 \\
&\qquad + 2\, \|BH(Z(t))\|_2 \|BZ\|_2 \\
&= 2\lambda_1 \|Z(t)\|_B^2 \\
&\qquad + 2\, \|H(Z(t))\|_B \|Z(t)\|_B \\
&\leq 2\lambda_1 \|Z(t)\|_B^2 \\
&\qquad + 2b\, \|H(Z(t))\|_2 \|Z\|_B
\end{aligned}
$$

$$\leq 2\lambda_1 \|Z(t)\|_B^2 + 2bC_\delta \|Z(t)\|_2^2 \|Z(t)\|_B$$

$$\leq 2\lambda_1 \|Z(t)\|_B^2 + 2bC_\delta \frac{1}{a^2} \|Z(t)\|_B^3$$

so... we have

$$\frac{d}{dt}\|Z(t)\|_B^2 \leq \underbrace{\left[2\lambda_1 + 2bC_\delta \frac{1}{a^2}\|Z(t)\|_B\right]}_{\substack{\lambda_1<0 \text{ and if we take } \delta \text{ small enough we} \\ \text{can make sure that} \\ Z(t) \in B_\delta(\vec{0}) \Rightarrow \\ 2bC_\delta \frac{1}{a^2}(b\delta) < -\lambda_1}} \|Z(t)\|_B^2$$

from $\|Z\|_2 < \delta \Rightarrow \|Z\|_B < b\delta$

$$\Rightarrow \frac{d}{dt}\|Z(t)\|_B^2 \leq \lambda_1 \|Z(t)\|_B^2$$

as desired.