MAT267: HW7

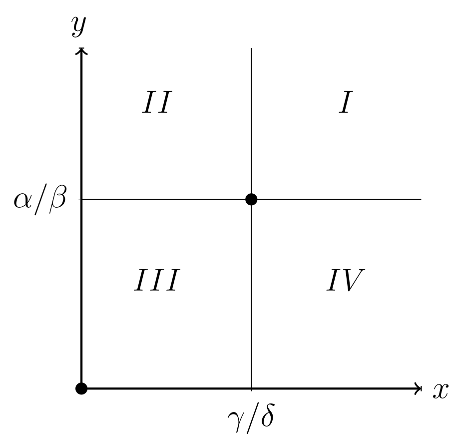
If you're choosing to do this assignment, please submit it by 11:59pm on Saturday April 25, 2020.

This document was updated April 24 to clarify problem 5b. Also the notation δ was overloaded in problems 1c-1f; this has now been fixed. Also the notation α and β was overloaded in problems 1g and 1j; this has now been fixed. And problem 4f was fixed.

1. (20 points) Recall the LotkaVolterra equations

$$\begin{cases} x' = \alpha x - \beta xy \\ y' = -\gamma y + \delta xy \end{cases}$$

where $\alpha, \beta, \gamma, \delta > 0$. We're interested in the dynamics in the $x, y \geq 0$ region of the plane. This twelve-part (!!) problem walks you through the steps to make the argument rigorous in the eyes of an analyst[1].



The four quadrants are the open sets

$$\begin{aligned} I &= \{(x,y) \,|\, x > \gamma/\delta, y > \alpha/\beta\} \\ II &= \{(x,y) \,|\, 0 < x < \gamma/\delta, y > \alpha/\beta\} \\ III &= \{(x,y) \,|\, 0 < x < \gamma/\delta, 0 < y < \alpha/\beta\} \\ IV &= \{(x,y) \,|\, x > \gamma/\delta, 0 < y < \alpha/\beta\} \end{aligned}$$

[1] The analyst would work through all of these steps on their own. To what degree they would share their calculations/arguments with others would depend on where they're planning to present the work. See the "AARGH!" in the photo of a journal article that Terry Tao posted in his tribute to Jean Bourgain `https://terrytao.wordpress.com/2018/12/29/jean-bourgain/`. As you've noticed, our textbook often presents proofs that rely on the reader to be able to identify where there are gaps that need to be filled in and to be able to figure out how to fill them in.

(a) Prove that the set $A_x := \{(x,0) \,|\, x > 0\}$ is invariant under the flow. Prove that if $X(t)$ is a solution of the nonlinear system and if there is some time t_0 so that $X(t_0) \in A_x$ then the maximal interval of existence of $X(t)$ is $(-\infty, \infty)$ and $X(t) \to \vec{0}$ as $t \to -\infty$ and $|X(t)| \to \infty$ as $t \to \infty$.

(b) Prove that the set $A_y := \{(0,y) \,|\, y > 0\}$ is invariant under the flow. Prove that if $X(t)$ is a solution of the nonlinear system and if there is some time t_0 so that $X(t_0) \in A_y$ then the maximal interval of existence of $X(t)$ is $(-\infty, \infty)$ and $X(t) \to \vec{0}$ as $t \to \infty$ and $|X(t)| \to \infty$ as $t \to -\infty$.

(c) Let $S_{I,II} = \{(\gamma/\delta, y) \,|\, y > \alpha/\beta\}$. Prove that if $X(t)$ is a solution of the nonlinear system and if there is some time t_0 so that $X(t_0) \in S_{I,II}$ then there is some $\widetilde{\delta} > 0$ so that $X(t) \in I$ for all $t \in (t_0 - \widetilde{\delta}, t_0)$ and $X(t) \in II$ for all $t \in (t_0, t_0 + \widetilde{\delta})$.

(d) Let $S_{II,III} = \{(x, \alpha/\beta) \,|\, 0 < x < \gamma/\delta\}$. Prove that if $X(t)$ is a solution of the nonlinear system and if there is some time t_0 so that $X(t_0) \in S_{II,III}$ then there is some $\widetilde{\delta} > 0$ so that $X(t) \in II$ for all $t \in (t_0 - \widetilde{\delta}, t_0)$ and $X(t) \in III$ for all $t \in (t_0, t_0 + \widetilde{\delta})$.

(e) Let $S_{III,IV} = \{(\gamma/\delta, y) \,|\, 0 < y < \alpha/\beta\}$. Prove that if $X(t)$ is a solution of the nonlinear system and if there is some time t_0 so that $X(t_0) \in S_{III,IV}$ then there is some $\widetilde{\delta} > 0$ so that $X(t) \in III$ for all $t \in (t_0 - \widetilde{\delta}, t_0)$ and $X(t) \in IV$ for all $t \in (t_0, t_0 + \widetilde{\delta})$. *Feel free to invoke your answer to part c) as much as possible, if you're confident in your answer to part c).*

(f) Let $S_{IV,I} = \{(x, \alpha/\beta) \,|\, x > \gamma/\delta\}$. Prove that if $X(t)$ is a solution of the nonlinear system and if there is some time t_0 so that $X(t_0) \in S_{IV,I}$ then there is some $\widetilde{\delta} > 0$ so that $X(t) \in IV$ for all $t \in (t_0 - \widetilde{\delta}, t_0)$ and $X(t) \in I$ for all $t \in (t_0, t_0 + \widetilde{\delta})$. *Feel free to invoke your answer to part d) as much as possible, if you're confident in your answer to part d).*

(g) Let $X(t) = (x(t), y(t))$ be a solution of the nonlinear system with maximal interval of existence $(\widetilde{\alpha}, \widetilde{\beta})$. Assume there is some time t_0 so that $X(t_0) \in I$. Prove that one of two things must happen: either $\widetilde{\beta} < \infty$ and $y(t) \to \infty$, or the solution crosses into region II at some time t_1 where $t_1 > t_0$.

(h) Let $X(t) = (x(t), y(t))$ be a solution of the nonlinear system. Assume there is some time t_0 so that $X(t_0) \in II$. Prove that the solution crosses into region III at some time t_1 where $t_1 > t_0$.

(i) Let $X(t) = (x(t), y(t))$ be a solution of the nonlinear system. Assume there is some time t_0 so that $X(t_0) \in III$. Prove that the solution crosses into region IV at some time t_1 where $t_1 > t_0$. *Feel free to invoke your answer to part g) as much as possible, if you're confident in your answer to part g).*

(j) Let $X(t) = (x(t), y(t))$ be a solution of the nonlinear system with maximal interval of existence $(\widetilde{\alpha}, \widetilde{\beta})$. Assume there is some time t_0 so that $X(t_0) \in IV$. Prove that one of two things must happen: either $\widetilde{\beta} < \infty$ and $x(t) \to \infty$, or the solution crosses into region I at some time t_1 where $t_1 > t_0$. *Feel free to invoke your* *answer to part h) as much as possible, if you're confident in your answer to part h).*

(k) Solve the separable ODE

$$\frac{dy}{dx} = \frac{-\gamma y + \delta xy}{\alpha x - \beta xy}$$

to implicitly determine a solution $(x, y(x))$ of the ODE in the first quadrant $\{(x, y) \mid x, y > 0\}$. Your solution $(x, y(x))$ satisfies an equation of the form $L(x, y(x)) = L(x_0, y_0)$. What is $L(x, y)$?

(l) If $(x_0, y_0) \neq (\gamma/\delta, \alpha/\beta)$, the level set of $L(x, y)$ through (x_0, y_0) is a closed curve in the first quadrant. This is something that you should (I hope!) know from MAT257. If you'd like to plot a few level sets, you can do this using wolframalpha. For example, `ContourPlot[{x+y==1,x+2y==1},{x,0,2},{y,0,2}]`

Prove that $L(x, y)$ is a Liapunov function for the Lotka-Volterra system on the open set $\mathcal{O} = \{(x, y) \mid x, y > 0\}$. (You may need to add a constant to L to ensure that $L(X^*) = 0$ at $X^* = (\gamma/\delta, \alpha/\beta)$ and $L > 0$ on $\mathcal{O} - X^*$.

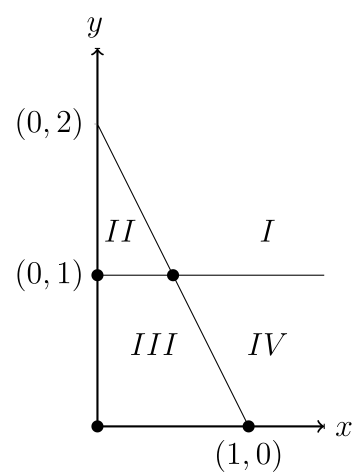
Now consider initial data $X_0 = (x_0, y_0)$ with $0 < x_0 \neq \gamma/\delta$ and $0 < y_0 \neq \alpha/\beta$. Prove that the solution of initial value problem with $X(t_0) = X_0$ has maximal interval of existence $(-\infty, \infty)$ and is a periodic solution whose solution travels in a counterclockwise manner in the first quadrant of the plane.

2. (15 points) Consider the nonlinear system

$$\begin{cases} x' = x\,(y + 2x - 2) \\ y' = y\,(y - 1) \end{cases} .$$

Let

$$III = \{(x, y) \mid 0 < x < 1 - y/2, 0 < y < 1\}.$$



Prove that if $X_0 \in III$ then the solution of the initial value problem with $X(t_0) = X_0$ has maximal interval of existence $(\alpha, \infty) \ni t_0$, that $X(t) \in III$ for all $t \geq t_0$, and $X(t) \to \vec{0}$ as $t \to \infty$.

3. (10 points) The Liapunov stability theorem is stated on page 193 and is proven on pages 196-197. The following is asking you to expand on two points that are not dwelled upon in the book.

   (a) Prove that there's guaranteed to be a sequence $\{t_n\}$ with $t_n \to \infty$ so that $\{X(t_n)\}$ converges to some point $Z_0 \in B_\delta(X^*)$?

   (b) Prove that if $X(t_n) \to X^*$ as $t_n \to \infty$ then $X(t) \to X^*$ as $t \to \infty$.

4. (15 points) Consider the system

$$\begin{cases} x' = y \\ y' = -x - 3\,\epsilon\, x^2 \end{cases}$$

Assume $\epsilon > 0$. We are interested in the behaviour of solutions in all of $\mathbb{R}^2$.

   (a) Plot the nullclines of the system and indicate the direction of the vector field along the nullclines. Indicate the regions in the phase plane where $x' > 0$, $x' < 0$, $y' > 0$, and $y' < 0$. Find the two equilibrium solutions of the system of ODEs: $X_{eq,1}$ and $X_{eq,2}$. Linearize the system at the equilibrium solutions and describe the behaviour of the nonlinear system near the equilibrium solutions.

   (b) Let

$$L(x, y) = y^2 + x^2 + 2\,\epsilon\, x^3.$$

   Verify that if $(x(t), y(t))$ is a solution of the system of ODEs then

$$\frac{d}{dt} L(x(t), y(t)) = 0$$

   and therefore solutions of the system lie in level sets of the function L.

   (c) Sketch some level sets of L. (Do this without using wolframalpha; you can figure out what they look like using symmetry arguments and arguments about cubics.) Make sure to include the level sets with values $L(X_{eq,1})$ and $L(X_{eq,2})$ as well as some values between these two values and some values larger than the maximum of the two values and some value smaller than the minimum of the two values. Label your level sets in such a way to make it clear which level sets correspond to which values.

   (d) Some level sets contain only one solution. That is, given a point on the level set, X_0, the resulting solution has maximal interval of existence $(-\infty, \infty)$ and

$$\{X(t) \,|\, t \in \mathbb{R}\} = \text{the entire level set}$$

   What are the level sets that contain only one solution? Put arrows on the level sets in your plot to indicate the direction of the motion of the solution.

   (e) Some level sets contain two solutions. You can find two points X_0 and Y_0 so that $X(t)$ and $Y(t)$ (the solutions through X_0 and Y_0 satisfy

$$\{X(t) \,|\, t \in \mathbb{R}\} \cup \{Y(t) \,|\, t \in \mathbb{R}\} = \text{the entire level set}$$

   What are the level sets that contain two solution? Put arrows on the level sets in your plot to indicate the direction of the motion of the two solutions.

(f) There is a level set that contains four solutions. You can find four points W_0, X_0, Y_0, and Z_0 so that $W(t)$, $X(t)$, $Y(t)$, and $Z(t)$ (the solutions through W_0, X_0, Y_0, and Z_0 satisfy

$$\{W(t) \mid t \in \mathbb{R}\} \cup \{X(t) \mid t \in \mathbb{R}\} \cup \{Y(t) \mid t \in \mathbb{R}\} \cup \{Z(t) \mid t \in \mathbb{R}\} = \text{the entire level set}$$

What is the level set that contains three solutions? Put arrows on the level set in your plot to indicate the direction of the motion of the three solutions.

5. (10 points) Given a system of ODEs, $X' = F(X)$, we can seek a Liapunov function L that will help us understand the behaviour of solutions, at least in regions of the phase space. (Whether or not we succeed, is a different question.) But we can also ask the opposite question: Given a function L that we want to minimize, can we try to minimize it by using it to create a system of ODEs, $X' = F(X)$, that we can study. In this problem, you'll do precisely that.

(a) Consider a function $L : \mathbb{R}^n \to \mathbb{R}$. If it's C^1, it has level sets and one can compute $\nabla L(x, y)$ to see the direction of fastest increase. This suggests a natural system of ODEs to consider: those given by *gradient descent*.

$$X' = F(X) := -\nabla L(X).$$

By construction, $\nabla L(X) \cdot F(X) \leq 0$ at all X. Now assume that L is C^2 so that we're guaranteed that $X' = F(X)$ has solutions and that they're unique.

Prove that if $X' = G(X)$ is some other C^1 vector field such at each point $\|F(X)\| = \|G(X)\|$ then L decreases more quickly along a solution $X(t)$ of $X' = F(X)$ than along a solution $Y(t)$ of $Y' = G(Y)$.

(b) Consider the function $L : \mathbb{R}^2 \to \mathbb{R}$ where

$$L(x, y) = 132 + 16\sqrt{3} - 10x - 4\sqrt{3}x + \frac{5}{4}x^2 + 28y + 2\sqrt{3}y - \frac{1}{2}\sqrt{3}xy + \frac{7}{4}y^2$$

Find and solve $X' = -\nabla L(X)$. As $t \to \infty$, what do solutions $X(t)$ converge to? What type of nonzero initial conditions yield solutions that converge the fastest to this point? What type of nonzero initial conditions yield solutions that converge the slowest to this point? Do some contour plots to come up with a conjecture as to why some initial conditions would lead to faster convergence and others lead to slower convergence.

(c) L is quadratic and so you can seek to write it in the form

$$L(x, y) = \begin{pmatrix} x - a & y - b \end{pmatrix} A \begin{pmatrix} x - a \\ y - b \end{pmatrix} = \begin{pmatrix} x - a & y - b \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x - a \\ y - b \end{pmatrix}$$

You could multiply the above out and match coefficients and solve for the six unknowns. Or you could use MAT257 methods to find the matrix A and the point (a, b). Explain how you would find A and the point (a, b). Having found A and point (a, b), are you confident that (a, b) is a global minimizer? How would you figure out whether or not it is? Do this for the matrix A that you found.

6. (10 points) You can do gradient descent on any C^2 function $L : \mathbb{R}^n \to \mathbb{R}$. Whether or not your solution goes to an equilbrium solution, whether or not the equilibrium solution is unique, and whether or not the equilibrium solution is a global (or even local) minimizer is a different question. Consider the function

$$L(x,y) = x^2\,y^2 \qquad \Longrightarrow \qquad \begin{cases} x' = -2\,x\,y^2 \\ y' = -2\,y\,x^2 \end{cases}$$

   (a) The definition of Liapunov function is given after the theorem on page 193. Why is L not a Liapunov function on $\mathbb{R}^2$?

   (b) In a similar manner as for the Lotka-Volterra system, you can find curves in the plane that solutions must live in. Given (x_0, y_0) with $x_0, y_0 > 0$, find $y(x)$ so that the solution of the IVP must lie in the graph of $y(x)$.

   (c) Prove that although L is not a Liapunov function, it is one when restricted to the graph of $y(x)$. From this, what must the solution of the initial value problem with $(x(t_0), y(t_0)) = (x_0, y_0)$ converge to as $t \to \infty$?

   (d) *To do if you're curious, no need to hand in.* You can find a solution of of the system that goes through (x_0, y_0) as follows. You seek a decreasing function $s(t)$ so that $(s(t), y(s(t)))$ lies in the graph $(x, y(x))$ and so that as $t \to \infty$ the solution $(s(t), y(s(t)))$ converges to the point you found in part c). How does the rate of convergence to the equilibrium solution depend on the initial point (x_0, y_0)? How does this compare to the situation where the function you're trying to minimize is a simple quadratic? (Or, more to the point, is a function that has local minima near which it's strictly convex.)

Obviously, the optimization problems in this homework are simple because you can find the critical points and the trajectories taken by gradient descent. In the real world, you'd be using a computer and would be hoping to find ways to do things faster than simple gradient descent.

There is an idea in the above that you may want to think about. In general, it can be hard to control solutions in $\mathbb{R}^2$ or in $\mathbb{R}^n$ because there's no ordering like there is in $\mathbb{R}$. A real-valued function that decreases along a solution imposes an ordering on a solution (see problem 3b and problem 6c). Similarly, nullclines divide the plane into regions in which one has an ordering on each component of the the solution as long as it is in the region in question.