Mat267: Midterm 2 March 9, 2020 5:10pm-7:00pm or 6:10-8:00pm

110 minute exam; please read all problems before starting. No calculators or other aids allowed.

The exam will be marked with the assistance of the crowdmark software. Do not write on the QR code at the top of the pages. Any page with a QR code will be scanned, uploaded, and read. Please write neatly and with a pen or dark pencil — light pencil writing doesn't get picked up well by the scanner.

Family name ....

Given name ....

Email ....**@mail.utoronto.ca**

UTORid ....

Signature:

SOLUTIONS

Make sure to read the back of this page!!

PLEASE DO NOT DETACH PAGES FROM THE EXAM.

If A is an $n \times n$ diagonalizable matrix then the general solution of $\vec{X}' = A\vec{X}$ is $\vec{X}(t) = \sum_{k=1}^{n} c_k \, e^{\lambda_k t} \vec{v}_k$ where $(\lambda_k, \vec{v}_k)$ are n eigenvalue-eigenvector pairs of A, chosen so that $\{\vec{v}_1, \vec{v}_k, \ldots \vec{v}_n\}$ is a linearly independent set.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = (ad - bc)\, I$$

The vectors E_1 and E_2 are

$$E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Equation-solving techniques

- To solve an equation of the form $\frac{dx}{dt} = f(t)\varphi(x)$, rewrite as $\frac{dx}{\varphi(x)} = f(t)dt$. Then take the integral. Don't forget the integration constant. In addition, do not forget to check the case when $\varphi(x) = 0$.
- To solve a first-order linear equation of the form $x' + p(t)x = q(t)$, first multiply the equation by $\mu(t)$. Then choose a $\mu(t)$ so that $\mu'(t) = \mu(t)p(t)$. With this choice of $\mu(t)$, the ODE becomes $(\mu(t)x(t))' = \mu(t)q(t)$. Integrate the equation with respect to t and, if possible, solve for $x(t)$.
- $X' = AX + G(t)$ with $X(t_0) = X_0$ has solution

$$X(t) = e^{tA} X_0 + \int_{t_0}^{t} e^{(t-s)A} G(s)\, ds$$

- $X' = A(t)X + G(t)$ with $X(t_0) = X_0$ has solution

$$X(t) = \Psi(t)\Psi(t_0)^{-1} X_0 + \int_{t_0}^{t} \Psi(t)\Psi(s)^{-1} G(s)\, ds$$

  where the columns of $\Psi(t)$ are n linearly independent solutions of $X' = A(t)X$.

- To solve $y'' + q(t)y' + r(t)y = g(t)$, first find linearly independent solutions y_1 and y_2 of $y'' + q(t)y' + r(t)y = 0$. A particular solution is $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ where

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W(t)}\, dt, \qquad u_2(t) = \int \frac{y_1(t)g(t)}{W(t)}\, dt$$

  Where

$$W(t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}.$$

1. a) (1 point) Consider the linear system $X' = AX$. What is the time-t map $\phi_t^A : \mathbb{R}^n \to \mathbb{R}^n$?

b) (2 points) The $n \times n$ matrices A and B are similar if there is an invertible matrix P so that

$$A = P\,B\,P^{-1}.$$

Assume A and B are similar matrices. What is the time-t map $\phi_t^B : \mathbb{R}^n \to \mathbb{R}^n$? How is it related to ϕ_t^A?

c) (5 points) Find a homeomorphism $H : \mathbb{R}^n \to \mathbb{R}^n$ so that $X' = AX$ and $X' = BX$ are conjugate linear systems. I.e. $H(\phi_t^A(X)) = \phi_t^B(H(X))$ for all $X \in \mathbb{R}^n$ and all $t \in \mathbb{R}$. Prove that your homeomorphism does what it needs to do and that it's invertible. (*Don't worry about proving that H and H^{-1} are continuous.*)

a) The time-t map $\phi_t^A = e^{tA}$

b) The time-t map $\phi_t^B = e^{tB}$ and $e^{tA} = Pe^{tB}P^{-1}$

c) Because $P^{-1}e^{tA} = e^{tB}P^{-1}$,

we have $P^{-1}e^{tA}X = e^{tB}P^{-1}X \quad \forall X \in \mathbb{R}^n$

Define $H : \mathbb{R}^n \to \mathbb{R}^n$ by $H(X) = P^{-1}X$.
Then H^{-1} is $H^{-1}(X) = PX$ because

$$H^{-1}(H(X)) = H^{-1}(P^{-1}X) = P(P^{-1}X) = X$$

<u>Note:</u>
$$\begin{aligned} H(\phi_t^A(X)) &= P^{-1}(e^{tA}X) \\ &= P^{-1}(Pe^{tB}P^{-1})X \\ &= e^{tB}P^{-1}X = \phi_t^B(H(X)) \quad \forall X \end{aligned}$$

as desired

2. (5 points) Consider the linear systems

$$X' = AX = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} X \qquad \text{and} \qquad X' = BX = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X$$

Prove that they are not conjugate linear systems.

$X' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X$ has general solution

$$X(t) = C_1 \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + C_2 \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$$

The second system has general solution

$$\tilde{X}(t) = C_1 \begin{pmatrix} \cos(2t) \\ 2\sin(2t) \end{pmatrix} + C_2 \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix}$$

The nonzero solutions of the first system have period 2π and the nonzero solutions of the second system have period π. Because $2\pi \neq \pi$, the systems cannot be conjugate. Why? Assume they are conjugate then $H(\phi_t^A(x)) = \phi_t^B(H(x))$. for some homeomorphism $H: \mathbb{R}^2 \to \mathbb{R}^2$.

$$H(\phi_\pi^A(x)) = H(\phi_0^A(x)) = H(x) \quad \forall x$$

Because $X' = AX$ is π-periodic.

If $H(\phi_\pi^A(x)) = \phi_\pi^B(H(x))$ then $H(x) = \phi_\pi^B(H(x))$ for all x. $\Rightarrow \phi_\pi^B(x) = x \ \forall x$ because H is invertible

$\therefore X' = BX$ has π-periodic solutions. $\times$

3. Consider the linear system

$$X' = AX = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ a & 0 & 0 \end{pmatrix} X$$

depending on the two parameters $a, b \in \mathbb{R}$.

a) (8 points) Find the general solution of this system. *It's fine if you find eigenvectors by inspection, but if you do this you need to demonstrate that they're actually eigenvectors.*

b) (2 points) Assume $a > 0$ and $b = 0$. Describe the behaviour of the solutions.

c) (2 points) Assume $a = 0$ and $b > 0$. Describe the behaviour of the solutions.

$$\det\begin{pmatrix} -\lambda & & \\ 0 & b-\lambda & 0 \\ a & 0 & -\lambda \end{pmatrix} = -\lambda \begin{vmatrix} b-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} + a \begin{vmatrix} a & 0 \end{vmatrix}$$

$$= -\lambda(b-\lambda)(-\lambda) + a(-a(b-\lambda))$$

$$= (b-\lambda)\left[\lambda^2 - a^2\right]$$

$$= (b-\lambda)(\lambda-a)(\lambda+a)$$

eigenvalues are $a, -a$, and b.

by inspection $\begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ a & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} \Rightarrow b, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

are an eigenpair

By inpection $\begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ a & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ a \end{pmatrix} \Rightarrow a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

are an eigenpair

By inspection $\begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ a & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -a \\ 0 \\ a \end{pmatrix} \Rightarrow -a, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

are an eigenpair

We have a basis of eigenvectors so the general solution is:

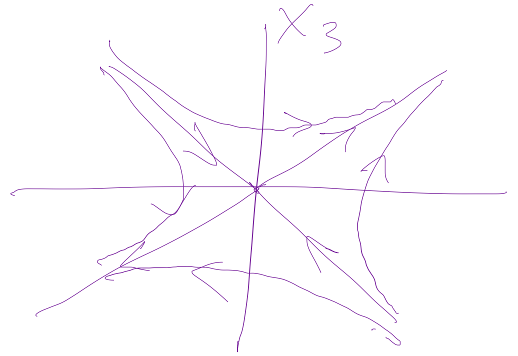
$$X(t) = c_1 e^{bt}\begin{pmatrix}0\\1\\0\end{pmatrix} + c_2 e^{at}\begin{pmatrix}1\\0\\1\end{pmatrix} + c_3 e^{-at}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

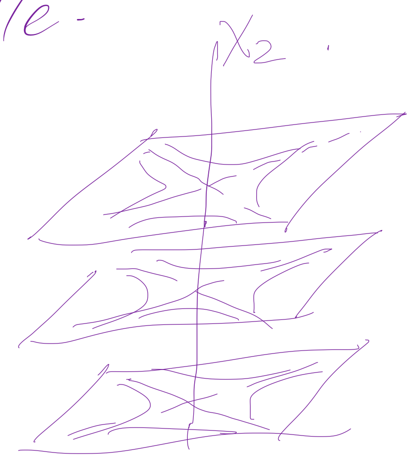
b) Assume $a>0$ and $b=0$

In this case

$$X(t) = \begin{pmatrix}0\\c_1\\0\end{pmatrix} + c_2 e^{at}\begin{pmatrix}1\\0\\1\end{pmatrix} + c_3 e^{-at}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

The second component is constant in time. In the x_1-x_3 plane there's a saddle.





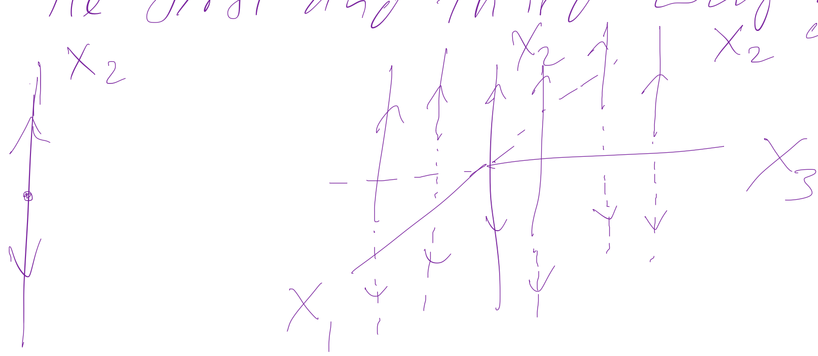
c) Assume $a=0$ and $b>0$.

In this case,

$$X(t) = c_1 e^{bt}\begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}c_2\\0\\c_3\end{pmatrix}$$

The first and third components are constant

X_2 comp has $|x_2(t)| \to \infty$ as $t \to \infty$



(my attempt at drawing lines piercing the X_1-X_3 plane)

4. a) (8 points) Find the general solution of

$$y'' + y' = \cos(t).$$

b) (2 points) Let $y(t)$ be a solution. There's a function $y_\infty(t)$ so that

$$\lim_{t\to\infty} |y(t) - y_\infty(t)| = 0.$$

Find $y_\infty(t)$ and demonstrate that the above limit is true.

You can find the solution using Variation of Parameters if you want. (See the formula sheet.) In which case you'd like to know that

$$\int e^{at}\cos(\omega t)\, dt = \frac{1}{a^2+\omega^2} e^{at}\left(a\cos(\omega t) + \omega \sin(\omega t)\right).$$

Or you can find the solution using some other method (in which case you'll need to demonstrate that what you found is the general solution).

Seek solution of $y'' + y' = 0$ of form $y(t) = e^{rt}$

$$r^2 e^{rt} + r e^{rt} = 0$$

$$r^2 + r = 0$$

$$r(r+1) = 0 \implies r = 0 \text{ or } r = 1$$

$$y_c(t) = C_1 e^{0t} + C_2 e^{-t} = C_1 + C_2 e^{-t}.$$

Use variation of parameters to find a particular solution $y_p(t)$.

$y_1(t) = 1 \qquad y_2(t) = e$

$y_1'(t) = 0 \qquad y_2'(t) = -e^{-t}$

$$W = \begin{vmatrix} 1 & e^{-t} \\ 0 & -e^{-t} \end{vmatrix} = -e^{-t}$$

$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$ where

$$u_1(t) = -\int \frac{y_2(t) g(t)}{W(t)} = -\int \frac{e^{-t}\cos(t)}{-e^{-t}}\, dt$$

$$= \int \cos(t)\, dt = \sin(t) + C$$

$$u_2(t) = \int \frac{y_1(t)g(t)}{W(t)} = \int \frac{1 \cos(t)}{e^{-t}}dt = -\int e^t \cos(t)\,dt$$

$$= -\frac{e^t}{2}(\cos(t) + \sin(t)) + \tilde{C}$$

$$y_p(t) = (\sin(t) + C)1 + \tfrac{1}{2}\left(-e^t(\cos(t) + \sin(t)) + \tilde{C}\right)e^{-t}$$

$$= \sin(t) + C - \frac{\cos(t)}{2} - \frac{\sin(t)}{2} + \tilde{C}e^{-t}$$

$$y_p(t) = C + \tilde{C}e^{-t} + \frac{\sin(t)}{2} - \frac{\cos(t)}{2}$$

This is the general solution, actually. The integration constants pick up the complementary solution $y_c(t)$.

$y_\infty(t) = C + \frac{\sin(t)}{2} - \frac{\cos(t)}{2}$ is a function such that $\lim_{t\to\infty} |y(t) - y_\infty(t)| = 0$.

$y = A\cos + B\sin$

$y' = -A\sin + B\cos$

$y'' = -A\cos - B\sin$

$y'' + y' = (B-A)\cos - (A+B)\sin = \cos.$

$\Rightarrow A + B = 0$

$B - A = 1$

$2B = 1 \Rightarrow B = \frac{1}{2}$

$\Rightarrow A = -\frac{1}{2}$

$y_p(t) = -\frac{1}{2}\cos(t) + \frac{1}{2}\sin(t)$ ✓

If I'd used the method of undetermined coeffs to find $y_p(t)$, it would have looked like this.

5. As you know, the Cayley-Hamilton Theorem states that a matrix A satisfies its own characteristic polynomial.

a) (2 points) Using this, what matrix equation does a 2×2 matrix A that has repeated eigenvalues λ and λ satisfy?

b) (5 points) Let V be a nonzero vector in $\mathbb{R}^2$. Show that either V is an eigenvector for A or $(A - \lambda I)V$ is an eigenvector for A.

a) $(A - \lambda I)^2 = A^2 - 2\lambda A + I = 0$

b) $(A - \lambda I)^2 = 0 \Rightarrow (A - \lambda I)(A - \lambda I)V = 0$

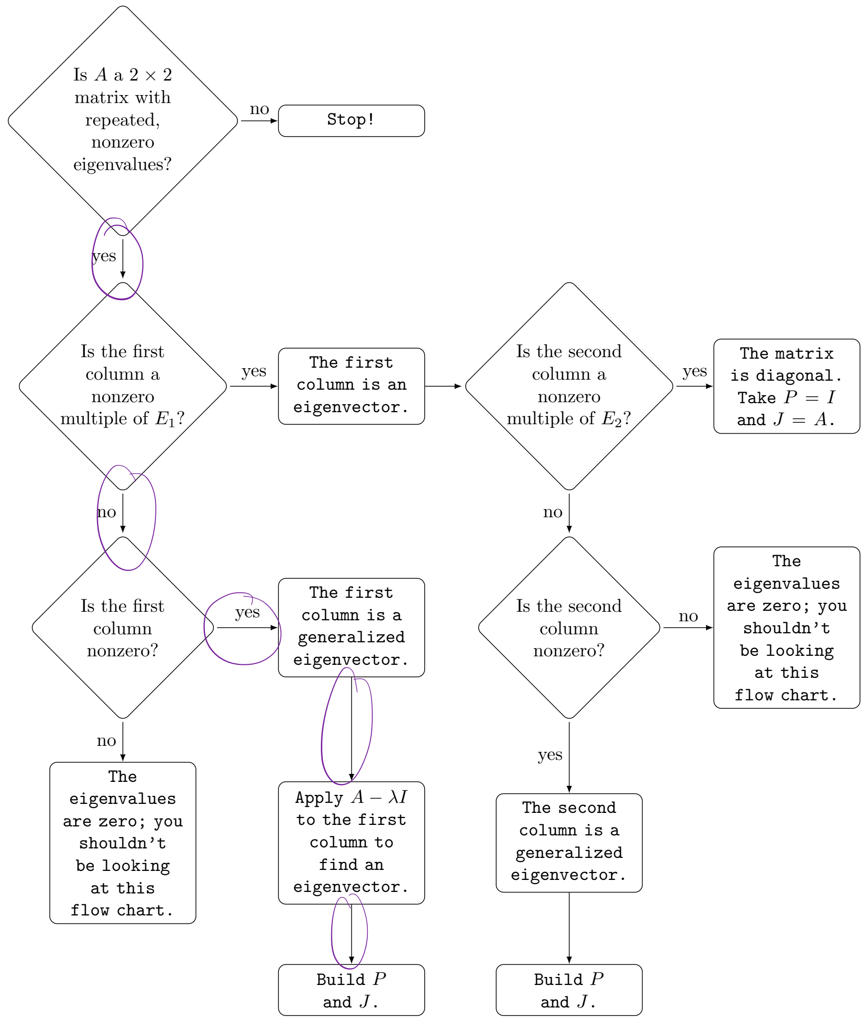
$\Rightarrow$ either $(A - \lambda I)V = 0$ in which case V is an eigenvector or

$(A - \lambda I)[(A - \lambda I)V] = 0$ where $(A - \lambda I)V \neq 0$

in which case $(A - \lambda I)V$ must be an eigenvector because $(A - \lambda I)[(A - \lambda I)V] = 0$.

$\Rightarrow (A - \lambda I)V$ is a generalized eigenvector.

c) Your cousin in Iceland sends you a fax about a new and improved way to diagonalize 2×2 matrices if the matrices have repeated, nonzero eigenvalues. *Don't dwell on the flow chart, please immediately read the stuff below it.*



Consider

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}$$

which has eigenvalues 4, 4.

i. (1 point) Find the path through the flow chart that you will need to follow for this matrix. Indicate the path by circling each arrow in the path.

ii. (5 points) Now implement your cousin's algorithm to find P and J so that $A = PJP^{-1}$. Did it work? *If you've blanked on how to get P^{-1} quickly for an invertible 2×2 matrix, see the formula sheet.*

$v_1 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $(A - 4I) = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$

and $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$

$\begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} -4 \\ -4 \end{pmatrix} = \begin{pmatrix} -16 \\ -16 \end{pmatrix} = 4\begin{pmatrix} -4 \\ -4 \end{pmatrix}$ ✓ it's an eigenvector

so v_1 is a generalized eigenvector.

Build J and P and check if they work.

$J = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$ $P = \begin{pmatrix} -4 & 3 \\ -4 & -1 \end{pmatrix}$

$P^{-1} = \frac{1}{16}\begin{pmatrix} -1 & -3 \\ 4 & -4 \end{pmatrix}$.

$$\frac{1}{16}\begin{pmatrix} -4 & 3 \\ -4 & -1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ 4 & -4 \end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix} -16 & 8 \\ -16 & -8 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ 4 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1/2 \\ -1 & -1/2 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ 4 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}$$

$= A$!! ☺

d) (5 points) Assume that A is a 2×2 matrix with repeated, nonzero eigenvalues. Assume that its first column, A_1, isn't a nonzero multiple of E_1. Prove that A_1 isn't an eigenvector (and must therefore be a generalized eigenvector).

$A_1 = AE_1$ and $A_1 \neq \begin{pmatrix} \mu \\ 0 \end{pmatrix}$ some $\mu \neq 0$.

1. A_1 isn't an eigenvector.

we know

$$AA_1 = A(AE_1) = A^2E_1$$

$$= [2\lambda A - I]E_1 \quad \text{(by the charact. polynomial)}$$

$$= 2\lambda AE_1 - E_1$$

$$= 2\lambda A_1 - E_1$$

A_1 isn't parallel to E_1 and so

$$2\lambda A_1 - E_1 \notin \text{span}\{A_1\}$$

$\Rightarrow AA_1 \notin \text{span}\{A_1\} \Rightarrow A_1$ isn't an eigenvector.

2. By part b)

if V isn't an eigenvector then it's a generalized eigenvector.