Full calculation for one example

$$A = \begin{pmatrix} 3 & 0 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \qquad \lambda = 2, 2, 2, 2$$

(1) Seek eigenvectors.

seek $V \neq \vec{0}$ so that $AV = 2V$

seek $V \neq 0$ so that $(A - 2I)V = 0$.

solve
$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} V = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

write augmented matrix and do elementary row operations:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

elementary row operations don't change the solution set, so we seek solutions of

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} V_1 + V_3 - V_4 &= 0 \\ V_2 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

V_3 and V_4 can be anything

$V_4 = s \in \mathbb{R}$

$V_3 = t \in \mathbb{R}$

$V_2 = 0$

$V_1 = V_4 - V_3 = s - t$

$$V = \begin{pmatrix} s-t \\ 0 \\ t \\ s \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

2 linearly independent eigenvectors are $V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and $V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

② Try to solve $(A-2I)X = V_1$ and

$(A-2I)X = V_2$

Write augmented matrix

$(A-2I \mid V_1 \mid V_2)$ and do

the elementary row operations and then extract the 2 simpler linear systems.

$$\left(\begin{array}{cccc|c|c} 1 & 0 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c|c} 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c|c} 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

So $(A-2I)X=V_1$ is equivalent to the system

$$\begin{aligned} x_1 + x_3 - x_4 &= 1 \\ x_2 &= 1 \\ 0 &= 1 \\ 0 &= 1 \end{aligned}$$

no solution! (we could have seen this from the beginning because row 4 of $A-2I$ is all zeroes and the 4th component of $V_1 \neq 0$.)

$(A-2I)X=V_2$ is equivalent to the system

$$\begin{aligned} x_1 + x_3 - x_4 &= -1 \\ x_2 &= -1 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

there are infinitely many solutions

$x_4 = s \in \mathbb{R}$

$x_3 = t \in \mathbb{R}$

$x_2 = -1$

$x_1 = -1 - x_3 + x_4 = -1 - t + s$

$(A-2I)X = V_2$ has general solution

$$X = \begin{pmatrix} -1-t+s \\ -1 \\ t \\ s \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + tV_2 + sV_1$$

yes those are the eigenvectors. Not surprising they're there. Think it through! ☺

we only need one solution. So choose one

$$U_1 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

(3) Solve $(A-2I)X = v_1$ or solve $(A-2I)^2 X = v_2$

the two problems are equivalent because $(A-2I)v_1 = v_2$. But I'll do the first problem because I don't need to compute $(A-2I)^2$

create augmented matrix $(A-2I \mid v_1)$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So $(A-2I)X = v_1$ is equivalent to this ↑ linear system

$$\begin{aligned} x_1 + x_3 - x_4 &= -1 \\ x_2 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

$x_4 = s \in \mathbb{R}$
$x_3 = t \in \mathbb{R}$
$x_2 = 0$
$x_1 = -1 - x_3 + x_4 = -1 - t + s$

Solution is
$$\begin{pmatrix} -1-t+s \\ 0 \\ t \\ s \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

again the general solution of $(A-2I)X = v_1$ is of the form

$$\begin{pmatrix}-1\\0\\0\\0\end{pmatrix} + tV_2 + sV_1$$

eigenvectors... no coincidence.

Need one solution, so choose one!

$$v_2 = \begin{pmatrix}-1\\0\\0\\0\end{pmatrix}$$

Ready to go! $P = \left(V_2 \middle| V_1 \middle| v_2 \middle| v_1\right)$

$$= \left(\begin{array}{c|c|c|c} -1 & -1 & -1 & 1\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{array}\right)$$

$$P = \begin{pmatrix}-1 & -1 & -1 & 1\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \qquad P^{-1} = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 1 & -1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 1 & -1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}3 & 0 & 1 & -1\\ 1 & 1 & 1 & -1\\ 0 & -1 & 2 & 0\\ 0 & 0 & 0 & 2\end{pmatrix}\begin{pmatrix}-1 & -1 & -1 & 1\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}2 & 1 & 0 & 0\\ 0 & 2 & 1 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 2\end{pmatrix} \quad \smile$$

Note: I found v_2 by solving $(A-2I)X = v_1$.
What if I'd found v_2 by solving

$$(A-2I)^2 X = V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \ ?$$

$$(A-2I)^2 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

build augmented matrix

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 &= -1 \\ 0 &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

is the linear system that's equivalent to $(A-2I)^2 = V_2$.

$x_4 = s$
$x_3 = t$
$x_2 = r$
$x_1 = -1 + r - t + s$

Solution is

$$\begin{pmatrix} -1+r-t+s \\ r \\ t \\ s \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

that is $(A-2I)^2 X = V_2$ has general solution

$$\begin{pmatrix}-1\\0\\0\\0\end{pmatrix} + r\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}-1\\0\\1\\0\end{pmatrix} + s\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

linear combination of eigenvectors.. this is in $\ker(A-2I)$

in $\ker((A-2I)^2)$ and is not in $\ker(A-2I)$

We need only one solution to choose v_2 and build the matrix P but we see that there are infinitely many choices determined up to the kernel of $(A-2I)^2$.

$A-2I$ has rank 2 $\Rightarrow \dim(\ker(A-2I)) = ②$
$\Rightarrow$ 2 eigenvectors & 2 jordan blocks.

$(A-2I)^2 = \begin{pmatrix}1 & -1 & 1 & -1\\0 & 0 & 0 & 0\\-1 & 1 & -1 & 1\\0 & 0 & 0 & 0\end{pmatrix}$ has rank 1

$\Rightarrow \dim(\ker((A-2I)^2)) = ③$

$\Rightarrow \exists$ 1 jordan block of size > 1.

↑ 1 because 3-2 = 1

$(A-2I)^3$ = zero matrix has rank 0

$\Rightarrow \dim(\ker((A-2I)^3)) = 4$

$\Rightarrow \exists$ 1 jordan block of size > 2

4

1 because $4-3=1$.

2 jordan blocks in a 4×4 matrix and one of them has size >2. Only one possibility:

$$A \sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

All of this numerology's fine and good but it doesn't help in constructing P so that

$$A = P \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} P^{-1}$$

Sometimes, things get a little tricky though!

In the previous example, I sought X so that

$$(A-2I)X = V_1 \text{ or } (A-2I)X = V_2$$

being able to solve this required that V_1 or V_2 is in the range of $(A-2I)$. But what if they aren't?

For example,

$$A = \begin{pmatrix} -10 & -5 & -11 & 16 \\ -6 & -5 & -9 & 12 \\ 0 & 0 & -2 & 0 \\ -7 & -4 & -19 & 12 \end{pmatrix}$$

the eigenvalues of A are $1, -2, -2, -2$

$$E_1 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$$E_{-2} = \text{span}\left\{\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

we need to find a vector v so that $(A-(-2)I)v$ is an eigenvector w/ eigenvalue -2

If we try to solve

$$(A-(-2)I)X = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

we find there's no solution. If we try to solve

$$(A-(-2)I)\ X = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

we find there's no solution.

This is because

$$\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \notin \text{range}\,(A-(-2)I)$$

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \notin \text{range}\,(A-(-2)I)$$

We need to find an eigenvector that <u>is</u> in the range of $A-(-4)I$

i.e. find α and β so that

$$\alpha\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \in \text{range}\,(A-(-2)I)$$

$$(A-(-2I)) = \begin{pmatrix} -8 & -5 & -11 & 16 \\ -6 & -3 & -9 & 12 \\ 0 & 0 & 0 & 0 \\ -7 & -4 & -10 & 14 \end{pmatrix}$$

Has reduced row echelon form

$$\begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ the range of $A-(-2I)$ is spanned by the first 2 columns of $A-(-2I)$

$$\text{range}(A-(-2)I) = \text{span}\left\{ \begin{pmatrix} -8 \\ -6 \\ 0 \\ -7 \end{pmatrix}, \begin{pmatrix} -5 \\ -3 \\ 0 \\ -4 \end{pmatrix} \right\}$$

So I seek α, β so that

$$\alpha \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \in \text{span}\left\{ \begin{pmatrix} -8 \\ -6 \\ 0 \\ -7 \end{pmatrix}, \begin{pmatrix} -5 \\ -3 \\ 0 \\ 4 \end{pmatrix} \right\}$$

$$\begin{pmatrix} -8 & -5 & -2 & 0 \\ -6 & -3 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ -7 & -4 & 0 & 1 \end{pmatrix} \text{ has RREF } \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \underbrace{-1 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}} = 1 \begin{pmatrix} -8 \\ -6 \\ 0 \\ -7 \end{pmatrix} - 2 \begin{pmatrix} -5 \\ -3 \\ 0 \\ -4 \end{pmatrix}$$

this is an eigenvector that __is__ in the range of $A-(-2)I$

So we choose a new basis for our eigen space

$$E_{-2} = \text{span}\left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

↑ $\vec{v}$ in Range $A-(-2)I$

$$\left(\begin{array}{cccc|c|c} -8 & -5 & -11 & 16 & 2 & 0 \\ -6 & -3 & -9 & 12 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -7 & -4 & -10 & 14 & 1 & 1 \end{array}\right)$$

has RREF
$$\left(\begin{array}{cccc|c|c} 1 & 0 & 2 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So we get

$$x_1 + 2x_3 - 2x_4 = 1$$
$$x_2 - x_3 = -2$$

$x_4 = t$

$x_3 = s$

$x_2 = -2 + s$

$x_1 = 1 - 2s + 2t$

$$\vec{u} = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} + \underbrace{s\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{eigenvectors}}$$

only need one choice for $\vec{u}$ so take

$$\vec{u} = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}.$$

$$P = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \qquad J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -2 & -1 & -3 & 4 \\ 2 & 1 & 2 & -3 \\ -1 & -1 & -1 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

done!