MATD01 Fields and Groups

Assignment 4

Hint for Problem 7

Hint for Parts (b) and (c): You can solve the problem directly if you want, that is by multiplying $(a + b\sqrt{-D})(c + d\sqrt{-D})$ and seeing when it can become 1 for Part (c) and 2 for Part (d) (with $D = 5$ for the latter). However, below is a neat way of doing it.

Define the *norm* function $N : \mathbb{Z}[\sqrt{-D}] \to \mathbb{Z}$ by

$$N(a + b\sqrt{-D}) = (a + b\sqrt{-D})(a - b\sqrt{-D}) = a^2 + Db^2.$$

A straightforward calculation shows that the norm is multiplicative, i.e. $N(\alpha\beta) = N(\alpha)N(\beta)$ for any $\alpha, \beta \in \mathbb{Z}[\sqrt{-D}]$. Now to find the units of $\mathbb{Z}[\sqrt{-D}]$, suppose $uu' = 1$. Then $N(u)N(u') = N(1) = 1$, so that $N(u) = N(u') = \pm 1$. Now you can easily see which elements of $\mathbb{Z}[\sqrt{-D}]$ have norm ± 1. (Remark: In what we said so far the fact that D is positive only matters in the last step, which is finding elements of norm ± 1. The situation for $D < 0$ is very different there. Read about Pell's equation if you are interested.)

For (d), follow the same idea: if 2 factors in $\mathbb{Z}[\sqrt{-5}]$ as $2 = \alpha\beta$, then apply the norm function to get $N(\alpha)N(\beta) = 4$. Now it should be easy to argue that α or β has to ± 1. (Note that if $b \neq 0$, then $N(a + b\sqrt{-5}) = a^2 + 5b^2 \geq 5$.)

The final remark in the question which asserts that 3 and $1 \pm \sqrt{-5}$ are also irreducible in $\mathbb{Z}[\sqrt{-5}]$ can similarly be verified.