University of Toronto Scarborough

MATD01 Fields and Groups

Takehome Final Exam

Due Wednesday April 15 at 12:00 pm
(to be submitted on Crowdmark)

Rules:

This assessment is subject to the University of Toronto Code of Behaviour on Academic Matters, available at: `https://governingcouncil.utoronto.ca/secretariat/policies/code-behaviour-academic-matters-july-1-2019`

It is your responsibility to be familiar and follow this code (in particular, see Section B1). In addition, this assessment is subject to the following rules:

- You may use your own notes and term work, as well as all official course material, i.e. the videos posted on Quercus and MATD01 channel on Youtube, the textbook, the previous assignments of our course, and their solutions posted on the course webpage.
- Any aid or resource (online or offline) other than those authorized above are not permitted.
- During the assessment period do not communicate about the questions or material on the assessment with any person other than the course instructor.
- Do not post the questions or answers to them anywhere online or otherwise.

Every single paper will be investigated, and cases of academic misconduct will be reported to the University. Every year a handful of students are reported to the Office of Academic Integrity and receive various penalties, which may range from an F in the course to expulsion from the University.

Instructions:

- You must complete and sign the Honour Pledge on the next page before you start working on the assessment. After you finish working on the assessment, you must complete and sign the Declaration Form provided on the last page of the assessment. The Honour Pledge and Declaration Form must be submitted (together with the rest of the assessment) before the deadline. The assessment will be considered as not submitted if the Honour Pledge or Declaration Form is missing (or is not completed and signed).
- The Honour Pledge, Declaration Form, and your answer to each question are to be uploaded separately on Crowdmark before the deadline. In case of technical issues, you must submit these by email to the course instructor before the deadline. If you are sending your solutions to the course instructor, please only attach one file per question (JPEG or PDF).

Honour Pledge

I pledge to honour myself and my community by assuring that the work I do on this assessment fully represents my own knowledge and ideas. I pledge to fully adhere to the University of Toronto Code of Behaviour on Academic Matters and the rules listed on the cover page of this assessment.

Full Name	Student Number
Signature	Date

Problems

Notes: There are 7 questions on this assessment. All claims must be justified. Answers without or with wrong justification will not be given any credit. In questions with multiple parts, to answer each part you may assume the results of the earlier parts (if they help), regardless of whether or not you have solved those earlier parts.

1. (4 points) Show that $x^9 - 3$ is irreducible over $\mathbb{Q}(e^{2\pi i/5})$.

2. (6 points) Let K be a splitting field of $f(x) = x^{15} - 2$ over $\mathbb{Q}$. Let $\alpha \in K$ be a root of $x^{15} - 2$ and $\zeta \in K$ a primitive 15th root of unity.

(a) (4 points) Show that $Gal(K/\mathbb{Q})$ is a group of order 120, generated by three elements δ, τ, and ϕ satisfying

$$\begin{aligned} \delta(\alpha) &= \alpha\zeta & \delta(\zeta) &= \zeta \\ \tau(\alpha) &= \alpha & \tau(\zeta) &= \zeta^2 \\ \phi(\alpha) &= \alpha & \phi(\zeta) &= \zeta^7. \end{aligned}$$

(b) (2 points) Show that K is not contained in any cyclotomic extension of $\mathbb{Q}$.

3. (5 points) Let p and q be distinct prime numbers. Show that the polynomial

$$f(x) = x^4 - 2(p+q)x^2 + (p-q)^2$$

is irreducible over $\mathbb{Q}$.

4. (4 points) Let $f \in \mathbb{Q}[x]$ be irreducible over $\mathbb{Q}$. Suppose

$$f(x) = \prod_{i=1}^{n}(x - \alpha_i)$$

in $\mathbb{C}[x]$. Show that

$$\sum_{\substack{1 \le i,j \le n \\ i \ne j}} \alpha_i^2 \alpha_j$$

is in $\mathbb{Q}$. (Hint: The solution is not long. One of the results from Assignment 9 is useful.)

5. (10 points) Let $\mathbb{F}_{81}$ be a field with 81 elements.

(a) (2 points) Show that $\mathbb{F}_{81}$ contains a primitive 16th root of unity.

(b) (4 points) Let $\beta \in \mathbb{F}_{81}$ be a primitive 16th root of unity. Let f_β be the minimal polynomial of β over $\mathbb{F}_3$. Show that

$$f_\beta(x) = (x - \beta)(x - \beta^3)(x - \beta^9)(x - \beta^{11}).$$

(Hint: What do we know about $Gal(\mathbb{F}_{81}/\mathbb{F}_3)$?)

(c) (4 points) Show that

$$\beta^2 + \beta^6 = \pm 1.$$

6. (16 points) Let $\zeta \in \mathbb{C}$ be a primitive n-th root of unity. Let $\lambda = \zeta + 1/\zeta$.

(a) (4 points) Show that $\mathbb{Q}(\zeta) \cap \mathbb{R} = \mathbb{Q}(\lambda)$.

(b) (4 points) Let f be the minimal polynomial of λ over $\mathbb{Q}$. Show that $\mathbb{Q}(\lambda)$ is a splitting field of f over $\mathbb{Q}$.

(c) (4 points) Show that

$$Gal(\mathbb{Q}(\lambda)/\mathbb{Q}) \simeq (\mathbb{Z}/n\mathbb{Z})^\times \,/\, \langle -1 \rangle.$$

(d) (4 points) Find the minimal polynomial of $2\cos(2\pi/9)$ over $\mathbb{Q}$.

7. (10 points) Give an outline of the proof of the following

THEOREM. There is a polynomial over $\mathbb{Q}$ which is not solvable by radicals.

An ideal outline has the following characteristics:

(1) It does not discuss proofs of statements that in spirit are more like exercises. For instance, you should not discuss why S_5 is not solvable, or include the calculation of the Galois group of any specific polynomial.
(2) It is not too technical (e.g. no actual proof of any statement needs to be there). The outline should more focus on the key ingredients and ideas of the proof, rather than the details of each step, especially for the more technical parts of the argument.
(3) It is clear and precise, and gives a mathematically strong audience enough information so that with some effort, they can complete the proof, or that at least, they get a fairly good idea about how the proof works.
(4) It is written for an audience that is not aware of any connection between properties of Galois groups and solvability of polynomials by radicals.
(5) It is a *maximum* of one page.

Aside from Item (4) above, there is some degree of freedom for what you can assume as the level of your audience. You may assume that your audience is familiar with all the relevant definitions and basic results, say to the extent of Chapter 12 of Rotman (Roots of Unity). On the other hand, if you prefer to recall certain definitions and results to make your outline accessible to a larger audience, feel free to do so.

Declaration Form

Congratulations - you have made it to the end of your assessment for this course! We hope that you feel proud of the work that you did here because you know that it was your own and no one else's. Please know that all suspected cases of academic dishonesty will be investigated following the procedures outlined in the Code of Behaviour on Academic Matters. If you have violated that Code or other rules of this assessment mentioned on the cover page, admitting it now will significantly reduce any penalty you incur. Admitting your mistakes is as much a matter of pride as never making them from the beginning. Thus, please check the appropriate statement below and complete the rest of the form.

☐ I confirm that the work I have done here is my own and no one else's, and is in line with the principles of scholarship and the University of Toronto Code of Behaviour on Academic Matters, and other rules given on the cover page of this assessment.

☐ I regret that I violated the Code of Behaviour on Academic Matters or other rules of this assessment and would like to admit that now so that I can take responsibility for my mistake.

I confirm that my response here is an accurate and true representation of my behaviour, knowing that by signing this declaration untruthfully I will incur an even greater penalty if it is later discovered that I have cheated or behaved dishonestly on this assessment.

Full Name ____________________ Student Number ____________________

Signature ____________________ Date ____________________