MAT301 Groups and Symmetry

Assignment 1

Due Friday September 21 at the beginning of the lecture

Please write your solutions neatly and clearly. Note that we may decide to grade only some of the questions (due to time limitations).

1. Determine if each of the following is a group.

(a) $\mathbb{Z}$ under $\star$ defined by $x \star y = x + y + xy$
(b) $\mathbb{Q} - \{-1\}$ under $\star$ defined by $x \star y = x + y + xy$ (First make sure that $\star$ is a binary operation on $\mathbb{Q} - \{-1\}$.)
(c) the set $\mathbb{R}_{>0}$ of positive real numbers under $\star$ defined by $x \star y = xy^2$ (So for instance, $2 \star 3 = 18$.)
(d) the set of all invertible 2×2 matrices with entries in $\mathbb{R}$ under matrix multiplication
(e) the set of all invertible 2×2 matrices with entries in $\mathbb{Z}$ under matrix multiplication

2. Suppose $(G, \star)$ is a group. Let $g, h, h' \in G$.

(a) Show that if $h \star g = h' \star g$, then $h = h'$. (In other words, "right cancellation" holds in a group. One can similarly show that "left cancellation" holds in a group as well, i.e. $g \star h = g \star h'$ implies $h = h'$.)
(b) Suppose $g \star h = h' \star g$. Does it follow that $h = h'$? Suggestion: Look for a counter-example in D_3 (the group of symmetries of an equilateral triangle, which you studied in your tutorial activity).
(c) Now suppose moreover that $(G, \star)$ is abelian. Does $g \star h = h' \star g$ imply $h = h'$?

3. Let $G = \{e, g\}$ be a group with two elements, with e the identity. Find the Cayley table of G (and provide full justification for your answer).

4. (a) Let G be a group. Let g be an element of G. Define a function $\phi_g : G \to G$ by $\phi_g(h) = gh$ (i.e ϕ_g sends every $h \in G$ to gh). Show that ϕ_g is a bijection.

(b) True or false: If G is a group, then every element of G appears in every row of the Cayley table of G exactly once.

5. Let G be a finite group. Denote the identity of G by e. Show that for every element $g \in G$, there is a positive integer n such that $g^n = e$. (In other words, show that every element of a finite group has finite order.)

6. Let G be a group with identity element denoted by e. Suppose G has the following property: for every $g \in G$, we have $g^2 = e$. Show that G is abelian. (Suggestion: Let $g, h \in G$. Start with $(gh)(gh) = e$. Now multiply both sides by h on the right. Be sure to carefully justify all steps of your calculation using group axioms.)

Practice Problems: The following problems are for your practice. They are not to be handed in for grading.

1. (a) Calculate the Cayley tables of each of the following groups: (i) The group D_3 of symmetries of an equilateral triangle (ii) The group of all rotational symmetries of a regular hexagon, denoted by R_6. (Use notation as in the tutorial worksheet. Denote counter-clockwise rotation by θ around the center by ρ_θ.) Note that R_6 has 6 elements.

(b) Find the order of every element of D_3 and R_6.

(c) Based on the data gathered in this problem, formulate a conjecture about a likely relation between the order of a finite group G and the order of each element of G. Try to make your conjecture as precise as you can.

2. Calculate the Cayley table of the group S_3 of all bijections $\{1, 2, 3\} \to \{1, 2, 3\}$ (under composition of functions). Find the order of every element of S_3.

3. Suppose G is a group with identity e. Let $g, h \in G$ satisfy $gh = e$. Show that $h = g^{-1}$. (So if we already know that G is a group, to show $h = g^{-1}$ it is not necessary to verify that both gh and hg are identity. It is enough to verify one of the two.)

4. Suppose $(G, \star)$ is a group of order 3. Let $G = \{e, g, h\}$, and e be the identity element.

(a) Show that that $g \star h = e$.
(b) What is $g \star g$? Can you determine the full Cayley table of G?
(c) Can we draw the following conclusion: On any set $S = \{x, y, z\}$ with three elements, there is a unique binary operation under which S is a group with identity element x.

5. Let G be a group. Let $g \in G$ and n be a positive integer.

(a) Explain what each of $(g^n)^{-1}$ and $(g^{-1})^n$ by definition means.
(b) Show that $(g^n)^{-1} = (g^{-1})^n$. (Remark: We define g^{-n} to be $(g^n)^{-1}$, or equivalently, $(g^{-1})^n$. So for instance, g^{-2} means either of $(g^2)^{-1}$ (i.e. the inverse of g^2) or $(g^{-1})^2$. The latter two elements are the same by the exercise, so there is no ambiguity.)
(c) By convention, define g^0 to be the identity element. Show that for every $m, n \in \mathbb{Z}$, $g^m g^n = g^{m+n}$.

6. Let G be a group. Define a function $\varphi : G \to G$ by $\varphi(g) = g^{-1}$ (i.e. φ maps every element to its inverse). Show that φ is a bijection.

7. (a) Let G be a group. Let g be an element of G. Define a function $\phi : G \to G$ by $\phi(h) = hg$ (i.e ϕ sends every $h \in G$ to hg). Show that ϕ is a bijection.

(b) Interpret the result in part (a) in terms of the Cayley table of G.

8. Let V be a real vector space. Let $GL(V)$ be the set of all isomorphisms (i.e. bijective linear transformations) $V \to V$. Show that $GL(V)$ is a group under the composition of functions. (This group is called the general linear group on V.)

9. Let G be a group. Denote the identity element of G by e.

(a) Is the subset

$$S = \{g \in G : g^2 = e\}$$

necessarily a subgroup? Prove or give a counter-example.

(b) Suppose moreover that G is abelian. Show that the set S of part (a) is a subgroup.