MAT301 Groups and Symmetry

Assignment 2

Due Friday Oct 5 at 11:59 pm
(to be submitted on Crowdmark)

Please write your solutions neatly and clearly. Note that we may decide to grade only some of the questions (due to time limitations).

1. In each part, a group G and a subset $S \subset G$ are given. Determine if S is a subgroup of G.

(a) $G = D_n$ (the group of symmetries of a regular n-gon), S = the set of all the rotational symmetries in D_n.
(b) $G = GL_n(\mathbb{R})$, $S \subset GL_n(\mathbb{R})$ the subset consisting of all the invertible diagonal matrices.
(c) $G = SL_2(\mathbb{Z})$ (the group of 2×2 matrices of determinant 1 which have integer entries), S the subset consisting of the matrices of the form $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, where $n \in \mathbb{Z}$.
(d) Fix an integer n. Let G be any abelian group with identity denoted by e, and $S = \{g \in G : g^n = e\}$.
(e) G any abelian group, S the set of all elements of finite order.
(f) $G = \mathbb{C}^\times$ (nonzero complex numbers under multiplication), S the unit circle in $\mathbb{C}$ (so $S = \{z \in \mathbb{C} : |z| = 1\}$, where $|z|$ means the norm of the complex number z).

2. (a) Give an example of a group G and elements $g, h \in G$ such that $|g|$ and $|h|$ are finite, but $|gh|$ is infinite. (Hint: Consider the group of symmetries of a circle.)

(b) Give an example of a group G in which the subset $S = \{g \in G : g^2 = e\}$ is not a subgroup.

3. For any group G, the *centre* of G (usually denoted by $Z(G)$) is defined as

$$Z(G) := \{g \in G : gh = hg \text{ for every } h \in G\}$$

(i.e. the set of those $g \in G$ which commute with every element of G).

(a) True or false: A group G is abelian if and only if $Z(G) = G$.
(b) Show that in general, $Z(G)$ is a subgroup of G.
(c) Find the centre of each of the groups D_n, S_n, and $GL_2(\mathbb{R})$. (For D_n and S_n assume $n \geq 3$.)

Suggestion: For D_n, it might be useful to think about the following question first: Let r be a reflection and ρ a rotation, with the centre of the rotation on the line of reflection. When is $r \circ \rho = \rho \circ r$? Think about the image of a point on the line of reflection under the two compositions. For $GL_2(\mathbb{R})$, it is easy to see that any matrix of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ commutes with every 2×2 matrix, so that there centre of $GL_2(\mathbb{R})$ certainly contains all such matrices (with $a \neq 0$, of course). Does the centre of $GL_2(\mathbb{R})$ contain any element besides these?)

4. Let G be a subgroup of S_n. Note that, in particular, each element of G is a bijection $\{1,\ldots,n\} \to \{1,\ldots,n\}$. Define a relation $\sim$ on the set $\{1,\ldots,n\}$ as follows: for any integers $1 \leq a, b \leq n$, set $a \sim b$ if and only if there exists $g \in G$ such that $g(a) = b$.

(a) Show that $\sim$ is an equivalence relation.
(b) Let $n = 5$ and $f \in S_5$ be the function that sends $1 \mapsto 2$, $2 \mapsto 4$, $3 \mapsto 5$, $4 \mapsto 1$, $5 \mapsto 3$. Let $G = \langle f \rangle$ (the subgroup of S_5 generated by f). Calculate the equivalence classes of the relation $\sim$ defined as above.

5. (a) Let G be a group. Let H be a subgroup of G. Define a relation $\sim$ on G as follows: for any $g, g' \in G$, set $g \sim g'$ if and only if there exists $h \in H$ such that $g' = gh$. Show that $\sim$ is an equivalence relation.

(b) Take $G = D_6$ and $H = \langle \rho_{2\pi/3} \rangle$, where ρ_θ denotes counter-clockwise rotation by θ. Calculate the equivalence classes of the relation $\sim$ defined as above.

6. Calculate the order of every element of each of the following groups: (a) $\mathbb{Z}/8$ (= residue classes mod 8 under addition) (b) μ_8 (= the subgroup of $\mathbb{C}^\times$ consisting of the 8th roots of unity) (c) $U(16)$ (Suggestion: Keep the formula $|g^k| = \frac{|g|}{\gcd(|g|,k)}$ in mind.)

Practice Problems: The following problems are for your practice. They are not to be handed in for grading.

0. Let G be an abelian group and $g, h \in G$ be of finite order. Show that $|gh| \leq \mathrm{lcm}(|g|, |h|)$, where lcm means the least common multiple.

1. Write the subgroup of $\mathbb{Q}$ generated by $\frac{1}{3}$ explicitly.

2. We say a group G is *cyclic* if there exists $g \in G$ such that $G = \langle g \rangle$. Which of the following groups are cyclic? (a) $\mathbb{Z}$ (b) $\mathbb{Z}/n$ (c) μ_n (d) $U(16)$ (e) D_n (f) $\mathbb{Q}$

3. (a) Show that the intersection of any nonempty collection of subgroups of a group G is a subgroup of G.
(b) Show that the union of two subgroups is a subgroup only if one of the two subgroups is contained in the other.

4. Let g be an element of a group G.

(a) Show that if a subgroup H of G contains g, then H contains $\langle g \rangle$.
(b) Show that $\langle g \rangle$ is the intersection of all subgroups of G which contain g. (This characterization of $\langle g \rangle$ is used in the next problem to define more generally the subgroup generated by any subset of a group.)

5. Let G be a group and S be a subset of G. Let $\langle S \rangle$ be the intersection of all the subgroups of G that contain S. (Note that there is at least one such subgroup, namely G itself.) Then $\langle S \rangle$ is a subgroup of G (by 3a); it is called *the subgroup generated by* S. If $S = \{g_1, \ldots, g_n\}$, we might write

$\langle g_1, \ldots, g_n \rangle$ instead of $\langle S \rangle$.

(a) Write the subgroup $\langle 2, 3 \rangle$ of $\mathbb{Q}^\times$ explicitly.
(b) We say a group G is *finitely generated* if there exist finitely many elements $g_1, \ldots, g_n \in G$ such that $G = \langle g_1, \ldots, g_n \rangle$. Show that $\mathbb{Q}^\times$ is not finitely generated.

6. Let r, r' be any two distinct reflections in D_3. Show that $D_3 = \langle r, r' \rangle$. (Larger dihedral groups are also generated by two reflections, but we cannot pick the reflections arbitrarily.)

7. (a) Let n be a positive integer. Define

$$\Gamma(n) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \ a, d \equiv 1 \pmod{n} \text{ and } b, c \equiv 0 \pmod{n}\}.$$

Show that $\Gamma(n)$ is a subgroup of $SL_2(\mathbb{Z})$.

8. True or false: If $K \leq H$ (i.e. if K is a subgroup of H) and $H \leq G$, then $K \leq G$.

9. Define

$$O_n(\mathbb{R}) := \{A \in GL_n(\mathbb{R}) : AA^T = I\}.$$

Show that $O_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$. (The group $O_n(\mathbb{R})$ is called the orthogonal group of degree n.)

10. Let V be a vector space and $\langle \, , \, \rangle$ be an inner product on V. Let $GL(V)$ denote the group of all isomorphisms $V \to V$ (under composition, see Question 8 of the practice list given in the first assignment). Let

$$H := \{g \in GL(V) : \langle g(v), g(w) \rangle = \langle v, w \rangle \text{ for all } v, w \in V\}.$$

(In words, H consists of those isomorphisms $V \to V$ which "preserve" $\langle \, , \, \rangle$.) Show that H is a subgroup of $GL(V)$.

11. Let H be a subgroup of G. Let $g \in G$. Define

$$gHg^{-1} := \{ghg^{-1} : h \in H\}.$$

Show that gHg^{-1} is also a subgroup of G.

12. Let S be a subset of a group G. The *centralizer* of S is defined to be

$$\{g \in G : gx = xg \text{ for every } x \in S\}$$

(i.e. the set of those $g \in G$ which commute with every element of S). Show that the centralizer of S is a subgroup of G.

13. (a) Prove Lagrange's theorem: if H is a subgroup of a finite group G, then $|H| \mid |G|$. (We will come back to this result in a few weeks and prove it in class, but I encourage the interested students to try to prove it themselves. The key is the equivalence relation defined in Question 5a of the assignment. How many elements does each equivalence class have?)

(b) Conclude that if G is a finite group, then the order of every $g \in G$ divides $|G|$. (Apply Lagrange's theorem to $H = \langle g \rangle$.)

14. Let H be a nonempty subset of a group G such that for every $g, h \in H$, we have $gh^{-1} \in H$. Show that H is a subgroup of G. (This result can help us save some time when we want to check that a given subset is a subgroup. Instead of checking closedness under the operation and taking inverses, we can just check that if $g, h \in H$, then $gh^{-1} \in H$.)

15. Let H be a nonempty subset of a finite group G. Suppose H is closed under the operation (i.e. if $g, h \in H$, then $gh \in H$). Show that H is a subgroup of G.