MAT301 Groups and Symmetry

Assignment 3

Due Friday Oct 26 at 11:59 pm
(to be submitted on Crowdmark)

Please write your solutions neatly and clearly. Note that we may decide to grade only some of the questions (due to time limitations).

1. Let G be a cyclic group of order 40. Let g be a generator of G.

(a) How many generators does G have? List them. No explanation is necessary.
(b) Does G have a subgroup of order 15? Explain.
(c) Does G have a element of order 12? Explain.
(d) How many subgroups of order 8 does G have? Explain.
(e) Find the subgroup H of G that has order 8 (i.e. give its elements). List all the generators of H. No explanation necessary.
(f) List all the elements of order 8 in G. No explanation necessary.

2. (a) Let n be a positive integer and G a cyclic group of order n. Let d be a positive divisor of n. Show that G has $\varphi(d)$ elements of order d.

(b) Show that $U(2^n)$ is not cyclic for $n \geq 3$. (Suggestion: Find the orders of $[2^{n-1}+1]$ and $[-1] = [2^n - 1]$. Is $[2^{n-1}+1] = [-1]$?)

3. (a) Let $m \mid n$. Show that $\mu_m \subset \mu_n$.

(b) As before, let $m \mid n$. Suppose H is a subgroup of μ_n of order m. What is H? Explain. (Do not do any computation.)

(c) Describe all subgroups of μ_n.

(d) Suppose K is a finite subgroup of $\mathbb{C}^\times$. Show that $K = \mu_m$ for some m.

4. (a) Prove that for any positive integer n,

$$\sum_{d \mid n} \varphi(d) \;=\; n.$$

(Here φ is Euler's function and the sum is over the positive divisors of n. Suggestion: Let G be a cyclic group of order n. For every $d \mid n$, let $\psi(d)$ denote the number of elements of G that have order d. Is it true that $n = \sum_{d \mid n} \psi(d)$?)

(b) Let K be a finite group with the following property: for every divisor $d \mid |K|$, there is a unique subgroup of order d in K. Show that K is cyclic.

Suggestion for (b): Let $|K| = n$. For every $d \mid n$, let $\psi(d)$ denote the number of elements of K that have order d. If you prove that $\psi(d) = \varphi(d)$ for every d (which devides n), you are done (why?). To show $\psi(d) = \varphi(d)$ for every d proceed as follows: (i) Try to argue that if $\psi(d) \neq 0$, that is, if K contains an element of order d, then $\psi(d) = \varphi(d)$. (ii) Argue that $n = \sum_{d \mid n} \psi(d)$. (iii) Combine (i) and (ii) and the formula in Part (a) to conclude that $\psi(d) = \varphi(d)$ for every d.

5. For each permutation σ given below, determine if σ is even or odd, write σ as a product (i.e. composition) of disjoint cycles, and find the order of σ.

(a) $\sigma = (1245)(245)(321)$
(b) $\sigma = (12)(123)(3214)^{-1}$
(c) $\sigma = (1245)^2$
(d) $\sigma = (1238)(457)(69)$
(e) $\sigma = (135)(246)(1265)(78)$
(f) $\sigma = (12)(23)(34)(45)$
(g) $\sigma = (15)(14)(13)(12)$

6. (a) Find all values that occur as the order of some element of S_6. (Your final list should include a number d if and only if there exists an element of order d in S_6.)

(b) Find the number of elements of S_6 of each order you listed in Part (a).

(c) Find the number of elements of A_6 that have order 4.

Practice Problems: The following problems are for your practice. They are not to be handed in for grading.

1. Show that any group of prime order is cyclic.

2. Let n be a positive integer. Let G be a group with finitely many elements of order n. Show that the number of elements of order n in G is a multiple of $\varphi(n)$.

3. Show that $\mathbb{Q}$ is not cyclic.

4. (a) Find all the possible values for the order of an element of S_7. (Your final list should include a number d if and only if there exists an element of order d in S_7.)

(b) Find the number of elements of S_7 of each order you listed in Part (a).

5. Let $f \in S_8$ be the function that sends $1 \mapsto 4$, $2 \mapsto 8$, $3 \mapsto 1$, $4 \mapsto 5$, $5 \mapsto 2$, $6 \mapsto 3$, $7 \mapsto 7$, and $8 \mapsto 6$.

(a) Write f as a product of disjoint cycles.
(b) Find the order of f.
(c) Write f^{-1} as a product of disjoint cycles.
(d) Write f as a product of transpositions.

6. Show that an ℓ-cycle is even if and only if ℓ is odd.

7. Determine if each of the permutations below is even or odd.

(a) (123)(134)(12)
(b) (123)(134)(12)(25)
(c) (12367)
(d) (12467)(234)(4568)
(e) (12467)(234)(4568)(12)

8. (a) Find all the possible values for the order of an element of A_8.

(b) Find the number of elements of A_7 of each order you listed in Part (a).