MAT301 Groups and Symmetry

Assignment 4

Due Friday Nov 9 at 11:59 pm
(to be submitted on Crowdmark)

Please write your solutions neatly and clearly. Note that due to time limitations, only some of the questions will be graded.

1. (a) Let $\phi : G \to H$ be a homomorphism. Let $g \in G$. Show that $\phi(\langle g \rangle) = \langle \phi(g) \rangle$ (i.e that the image of the subgroup generated by g under ϕ is the subgroup generated by $\phi(g)$).

(b) Conclude that the image of a cyclic group under a homomophism is cyclic (i.e. that if G is cyclic and $\phi : G \to H$ is a homomorphism, then $\phi(G)$ is cyclic).

(c) Let G and H be isomorphic groups. Show that G is cyclic if and only if H is cyclic.

2. (a) For any two groups G and H, we denote the set of all homomorphisms $G \to H$ by $\mathrm{Hom}(G, H)$. Show that for any group H, there is a bijective function

$$\mathrm{Hom}(\mathbb{Z}, H) \to H.$$

(Suggestion: Define $F : \mathrm{Hom}(\mathbb{Z}, H) \to H$ by $F(\phi) = \phi(1)$. Thus F sends a homomorphism $\phi : \mathbb{Z} \to H$ simply to $\phi(1) \in H$. Show that F is a bijection. Injectivity of F is the statement that if ϕ and ψ are homomorphisms $\mathbb{Z} \to H$ and $\phi(1) = \psi(1)$, then $\phi = \psi$ (i.e., that a homomorphism $\mathbb{Z} \to H$ is *determined* by its value at 1.) Surjectivity of F is the statement that for any $h \in H$, there is a homomorphism $\phi : \mathbb{Z} \to H$ such that $\phi(1) = h$.)

(b) List all homomorphisms $\mathbb{Z} \to S_3$.

3. Let G and H be finite groups such that $|G|$ and $|H|$ are relatively prime. Show that the only homomorphism $G \to H$ is the trivial map. In other words, show that if $\phi : G \to H$ is a homomorphism, then $\phi(g) = e$ for every $g \in G$. (Suggestion: Use Lagrange's theorem and the fact that $|\phi(g)| \mid |g|$.)

4. Show that the only homomorphism $\mathbb{Q} \to \mathbb{Q}^\times$ is the trivial map.

5. (a) Let G be a group. We say a subgroup $K \leq G$ is *normal* if for every $k \in K$ and $g \in G$, the element gkg^{-1} is in K. Let $\phi : G \to H$ be a homomorphism. Show that the subgroup $\ker(\phi)$ of G is a normal subgroup. (You don't have to rewrite the proof of the fact that the kernel is a subgroup; just verify normality.)

(b) Is there a homomorphism with domain S_{10} whose kernel is $\{e, (12)\}$? (Justify your answer.)

6. Determine which of the following groups are isomorphic: $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^\times$, $\mathbb{R}_{>0}$ (under multiplication), $\mathbb{C}^\times$, S_3, μ_6, $U(9)$, D_4, $\mathbb{Z}/8$, $U(16)$.

You may take the following fact for granted: two cyclic groups of the same order are isomorphic. (We will prove this in the next lecture.)

Practice Problems: The following problems are for your practice. They are not to be handed in for grading.

1. Let G be a group. Show that the map $\phi : G \to G$ defined by $\phi(g) = g^{-1}$ is a homomorphism if and only if G is abelian.
2. Let $\phi : G \to H$ be a homomorphism. Let $K \leq H$. Show that the preimage of K under ϕ, i.e. the subset

$$\{g \in G : \phi(g) \in K\}$$

is a subgroup of G. (The preimage of K under ϕ is usually denoted by $\phi^{-1}(K)$. Warning: Here ϕ^{-1} does not refer to the inverse function; the function ϕ need not be bijective and so there may be no inverse function.)
3. Let $\phi : G \to H$ be a homomorphism. Show that if G is abelian, then so is $\mathrm{Im}(\phi)$.
4. Let $\phi : G \to H$ be an injective homomorphism. Show that if H is abelian (resp. cyclic), then G is abelian (resp. cyclic).
5. Let $\phi : G \to H$ be a homomorphism.

(a) Show that if $g \in G$ has finite order, then $\phi(g)$ has finite order and $|\phi(g)| \mid |g|$. (We proved this in class, but it is useful to go over the proof one more time.)
(b) Suppose ϕ is injective. Show that for every element $g \in G$, we have $|g| = |\phi(g)|$.

6. Let G be a finite group of odd order. Show that the only homomorphism $G \to \mathbb{R}^\times$ is the trivial one.
7. Let G be a finite group and $\phi : G \to H$ a surjective homomorphism. Show that if H contains an element of order n, then do does G.
8. First, some notation: Given a group G and elements $g_1, g_2 \in G$, we denote the element $g_1 g_2 g_1^{-1} g_2^{-1}$ by $[g_1, g_2]$. The *commutator* (or *derived*) subgroup of G, denoted by $[G, G]$, is the subgroup of G generated by all the elements of the form $[g_1, g_2]$, where $g_1, g_2 \in G$. In other words, $[G, G]$ is the subgroup of G with the following two properties: (i) It contains $[g_1, g_2]$ for every $g_1, g_2 \in G$, and (ii) if K is any subgroup of G that contains all the elements of the form $[g_1, g_2]$, then K contains $[G, G]$.

Now let G and H be any groups and $\phi : G \to H$ a homomorphism. Show that $\mathrm{Im}(\phi)$ is abelian if and only if $[G, G] \subset \ker(\phi)$.
9. Show that the commutator $[G, G]$ is a normal subgroup of G.
10. Let G be a group. Show that $Z(G)$ (i.e. the centre of G) is a normal subgroup of G. (We already know $Z(G)$ is a subgroup; just prove normality.)
11. Let H be a subgroup of G. For every $g \in G$, let

$$gHg^{-1} := \{ghg^{-1} : h \in H\}.$$

Show that the following two statements are equivalent:

(i) For every $g \in G$, we have $gHg^{-1} \subset H$.
(ii) For every $g \in G$, we have $gHg^{-1} = H$.

(We gave (i) as the definition of normality. Some references give the seemingly more restrictive statement (ii) as the definition of normality. But the two statements are equivalent, as you show here.)
12. (a) Is there a homomorphism $\phi : \mathbb{Z}/12 \to S_7$ such that $\phi([1]) = (12345)$? If there isn't, prove so. If there is, define one.

(b) Is there a homomorphism $\phi : \mathbb{Z}/12 \to S_7$ such that $\phi([1]) = (1234)$? If there isn't, prove so. If there is, define one.

13. Let C_n be a cyclic group of order n. Let g be a generator of C_n. Let H be any group. Define

$$H[n] := \{h \in H : h^n = e\}.$$

(a) True or false: if $\phi : C_n \to H$ is a homomorphism, then $\phi(g) \in H[n]$.
(b) Show that $\phi \mapsto \phi(g)$ gives a bijection $\mathrm{Hom}(C_n, H) \to H[n]$.

14. Find all homomorphisms $\mathbb{Z}/6 \to \mathbb{Q}$.
15. Find all homomorphisms $\mathbb{Z}/6 \to \mathbb{C}^\times$.
16. Find all homomorphisms $\mathbb{Z}/n \to \mathbb{C}^\times$.
17. (a) Find all homomorphisms $S_3 \to \mathbb{C}^\times$. (Suggestion: We know two such homomorphisms, namely the trivial map and the sign function. Try to show that these are the only ones. For this, first find the commutator subgroup of S_3. The reason it is helpful to look at the commutator is that given any homomorphism $\phi : S_3 \to \mathbb{C}^\times$, image of ϕ (being a subgroup of $\mathbb{C}^\times$) is abelian, so that by Problem 8 above, the commutator of S_3 is contained in the kernel of ϕ.)
18. (challenge question, won't be on the test or exam.) (a) Show that every element of A_n can be written as a product of 3-cycles. (It might be helpful to calculate $(123)(134)(132)(143)$.)
(b) Show that the commutator subgroup of S_n is A_n.
(c) Find all homomorphisms $S_n \to \mathbb{C}^\times$.
19. We say a group G is divisible if it satisfies the following two properties: (i) G is abelian, and (ii) for every positive integer n and every $y \in G$, there exists an $x \in G$ such that $x^n = y$.

(a) Is each of the following groups divisible: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{R}^\times$, $\mathbb{R}_{>0}$, a finite group with more than one element, $S = \{z \in \mathbb{C}^\times : |z| = 1\}$ (the unit circle in $\mathbb{C}$ under multiplication), the subgroup of $\mathbb{C}^\times$ consisting of all the elements of finite order ($\cup_{n=1}^{\infty} \mu_n$).
(b) Let G be a divisible group and $\phi : G \to H$ be a homomorphism. Show that $\mathrm{Im}(\phi)$ is divisible.
(c) Is there any surjective homomorphism $\mathbb{C}^\times \to \mathbb{R}^\times$?

20. Let G be a group of order n. Let m be an integer relatively prime to n. Define $\phi : G \to G$ by $\phi(g) = g^m$.

(a) Show that ϕ is a bijection. (I suggest that you give a proof of this first assuming that G is abelian. For a general argument that also covers the nonabelian case, you need to use Lagrange's theorem and the fact that since $\gcd(m, n) = 1$, there exists an integer k such that $mk \equiv 1 \pmod n$.)
(b) Conclude that if G is abelian, the ϕ is an isomorphism. (An isomorphism of a group onto itself is called an *automorphism* of the group.)

21. Let $\phi : G \to H$ and $\psi : H \to K$ be homomorphisms.

(a) Show that $\ker(\phi) \subset \ker(\psi \circ \phi)$.
(b) Show that if ψ is injective, then $\ker(\phi) = \ker(\psi \circ \phi)$.

22. Let $\phi : G \to H$ be an isomorphism.

(a) Use ϕ to define a bijection between the set of subgroups of G and the set of subgroups of H.
(b) Show that for any subgroup K of G, K is normal in G if and only if $\phi(K)$ is normal in H.

23. Let $\phi : G \to H$ be an injective homomorphism. Show that G is isomorphic to a subgroup of H.
24. Find all isomorphic groups among D_3, S_3, μ_6, $U(9)$, $GL_2(\mathbb{R})$, $\mathbb{R}^\times$, D_4, $U(15)$, $\mathbb{Z}/8$.
25. Let G be a group. Fix $h \in G$. Define $\phi : G \to G$ be $\phi(g) = hgh^{-1}$. Show that ϕ is an isomorphism (so is an automorphism of G). (The map ϕ defined here is called *conjugation by* h.)

26. Let G be a group. Denote the set of all automorphisms of G (i.e. isomorphisms $G \to G$) by $\mathrm{Aut}(G)$.

(a) Show that $\mathrm{Aut}(G)$ is a group under composition of functions.

(b) For any $h \in G$, let ϕ_h be the automorphism of G defined by $g \mapsto hgh^{-1}$ (the map of the previous problem). Define a function

$$\Psi : G \to \mathrm{Aut}(G)$$

by $\Psi(h) = \phi_h$. Show that Ψ is a homomorphism with kernel equal to $Z(G)$ (= the centre of G). Terminology: Automorphisms in the image of Ψ (i.e. automorphisms that are given by conjugation by some element h) are called the *inner* automorphisms of G.

27. Define a nontrivial automorphism of $\mathbb{C}^\times$ (nontrivial means not equal to the identity map). Hint: Think about complex conjugation.

28. Can you give an automorphism of $GL_n(\mathbb{R})$ that is not an inner automorphism?

29. (a) Show that $\mathrm{Aut}(\mathbb{Z}) \simeq \mu_2$.

(b) Show that $\mathrm{Aut}(\mathbb{Z}/n) \simeq U(n)$. (This will not be on Test 2.)

30. Let $\phi : G \to H$ be an isomorphism. Let K be an arbitrary group. Use ϕ to define bijections

$$\mathrm{Hom}(G, K) \to \mathrm{Hom}(H, K)$$

and

$$\mathrm{Hom}(K, G) \to \mathrm{Hom}(K, H).$$

31. Let G and H be isomorphic groups. Show that $\mathrm{Aut}(G) \simeq \mathrm{Aut}(H)$.

32. Let G and H be groups. Given any two functions $\phi, \psi : G \to H$, define a function $\phi\psi : G \to H$ by

$$(\phi\psi)(g) = \phi(g)\psi(g).$$

(In other words, $\phi\psi$ is the function whose values are, element-wise, the product of values of ϕ and ψ.)

Now suppose H is abelian.

(a) Show that if ϕ and ψ are homomorphisms, then $\phi\psi$ is a homomorphism. (The assumption that H is abelian must be used in your argument, otherwise something is wrong.)

(b) Show that $\mathrm{Hom}(G, H)$ with the operation defined above is an abelian group.

(c) Show that the bijection $\mathrm{Hom}(\mathbb{Z}, H) \to H$ of Problem 2 of the assignment is also a homomorphism (so an isomorphism of groups).

33. Let G be a finite abelian group.

(a) Show that $\mathrm{Hom}(G, \mathbb{C}^\times)$ is a finite set. (In other words, show that there are only finitely many homomorphisms $G \to \mathbb{C}^\times$.) Terminology: Homomorphisms $G \to \mathbb{C}^\times$ are called *characters* of G.

(b) (Challenge question, won't be on the test or exam.) Fix $g \in G$. This question concerns the sum

$$\sum_{\chi \in \mathrm{Hom}(G, \mathbb{C}^\times)} \chi(g).$$

Here the sum is over all the characters of G; there are finitely many of them so this is a finite sum and makes sense. Show that if $g \neq e$, then

$$\sum_{\chi \in \mathrm{Hom}(G, \mathbb{C}^\times)} \chi(g) = 0.$$

You may take the following fact for granted: For any element $g \neq e$ of G, there is a character χ' of G such that $\chi'(g) \neq 1$.