MAT301 Groups and Symmetry

Assignment 6

Due Wednesday Dec 5 at 11:59 pm
(to be submitted on Crowdmark)

Please write your solutions neatly and clearly. Note that due to time limitations, only some of the questions will be graded.

1. The two parts of this question are not related to one another.

(a) Let G be a group and H a normal subgroup of G. Suppose the index $[G : H]$ is finite. Let $g \in G$ be an element of finite order such that $\gcd(|g|, [G : H]) = 1$. Show that $g \in H$.

(b) Let G be a group and $H \leq G$ a subgroup of index 2. Show that if $g, g' \in G - H$, then $gg' \in H$.

2. In each part, show that the given two groups are isomorphic.

(a) $GL_n(\mathbb{Q})/SL_n(\mathbb{Q})$ and $\mathbb{Q}^\times$

(b) $\mathbb{C}/\mathbb{Z}$ and $\mathbb{C}^\times$

(c) $\mathbb{R}/\mathbb{Z}$ and the unit circle $S = \{z \in \mathbb{C}^\times : |z| = 1\}$ (the latter under multiplication)

(d) $\mathbb{C}^\times/\mathbb{R}_{>0}$ and the unit circle (defined above)

(e) $\mathbb{C}^\times/\mathbb{R}^\times$ and the unit circle S

3. (a) Give a complete list of abelian groups of order 32, up to isomorphism. (Your list must contain exactly one group from each isomorphism class. In other words, your list must be such that every abelian group of order 32 is isomorphic to exactly one group from the list.)

(b) Find the order of [3] in $U(64)$. (The computations for this should not be difficult at all.)

(c) Show that $U(64)$ is isomorphic to $\mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

4. Throughout this question, we use the following notation: given integers a and n, we denote the residue class of a mod n by $[a]_n$.

(a) Let m and n be positive integers. Show that we have a well-defined homomorphism

$$\psi : \mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$$

given by $\psi([a]_{mn}) = [a]_m$.

(b) Prove the *Chinese remainder theorem*: If $\gcd(m, n) = 1$, then the map

$$\Phi : \mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$$

given by $\Phi([a]_{mn}) = ([a]_m, [a]_n)$ is an isomorphism. (You may use the fact that when m and n are relatively prime, a number is divisible by both m and n if and only if it is divisible by mn.)

(c) true or false (no explanation necessary): If m and n are positive integers with $\gcd(m, n) = 1$, then for every integers a and b, there exists an integer x which satisfies the system of equations

$$x \equiv a \pmod{m}$$
$$x \equiv b \pmod{n}.$$

Moreover, x is unique mod mn.

5. Let G_1, G_2 and H be abelian groups. Construct a bijective function

$$\mathrm{Hom}(G_1 \times G_2\,, H) \longrightarrow \mathrm{Hom}(G_1, H) \times \mathrm{Hom}(G_2, H)$$

(and prove that your function is bijective).

6. Give an example of an abelian group G and a subgroup $H \leq G$ such that G is not isomorphic to $H \times (G/H)$. Give two examples, one with G finite and one with G infinite.

Practice Problems: The following problems are for your practice. They are not to be handed in for grading.

1. Determine if there exists a homomorphism as described in each part below.

(a)-(f) as in Question 19 of the last set of practice problems

(g) a surjective homomorphism $U(80) \to D_{16}$

2. Let $\phi : U(125) \to U(25)$ be given by $\phi([a]_{125}) = [a]_{25}$.

(a) Is ϕ well-defined?

(b) Show that ϕ is a homomorphism.

(c) Find $\ker(\phi)$.

(d) Find the order of the element $[32]_{125} \cdot \ker(\phi)$ of the quotient group $U(125)/\ker(\phi)$.

(e) A student writes the following as solution to part (d). Is the argument correct?

"The order of $[32]_{125} \cdot \ker(\phi)$ *in the quotient* $U(125)/\ker(\phi)$ *is the same as the order of* $\phi([32]_{125}) = [32]_{25} = [7]_{25}$ *in* $U(25)$. *We have* $7^2 = 49 \equiv -1 \pmod{25}$, *and* $7^4 \equiv (-1)^2 \equiv 1 \pmod{25}$. *Thus the order of* $[7]_{25}$ *in* $U(25)$ *(and hence the order of* $[32]_{125} \cdot \ker(\phi)$ *in the quotient* $U(125)/\ker(\phi)$*) is equal to 4."*

3. Let n be a positive integer.

(a) Show that $\mathbb{C}^\times/\mu_n$ is isomorphic to $\mathbb{C}^\times$.

(b) Let k be a positive integer. How many elements of order k does $\mathbb{C}^\times/\mu_n$ have?

4. (a) Let $\phi : G \to H$ be a homomorphism. Let $h \in \mathrm{Im}(\phi)$. Show that the pre-image

$$\phi^{-1}(h) := \{g \in G : \phi(g) = h\}$$

is a coset of $\ker(\phi)$.

(b) Let G and H be groups, $|G| = 45$, $|H| = 15$, and $\phi : G \to H$ a surjective homomorphism. Let $h \in H$. Find the number of elements of the pre-image $\phi^{-1}(h)$.

5. (a) Give a complete list of abelian groups of order 160 up to isomorphism.

(b) Suppose G is an abelian group of order 160 which is not cyclic and that has an element of order 16. Show that G is isomorphic to $\mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$.

6. (a) Give a complete list of abelian groups of order 36 up to isomorphism.

(b) Let G be an abelian group of order 36 which does not contain any element of order 4. How many elements of G have order 2?

7. Question 32 of the last set of practice problems

8. Determine if each statement is true or false.

(a) If G is an abelian group of order 240, then the kernel of the map $\phi : G \to G$ given by $\phi(g) = g^{16}$ has 16 elements.

(b) If G is an abelian group of order 240, then the kernel of the map $\phi : G \to G$ given by $\phi(g) = g^8$ has 8 elements.

(c) If G is a group of order 240, then the set $\{g \in G : g^{16} = e\}$ has 16 elements.

(d) Every abelian group of order 30 is cyclic.

(e) If G is an abelian group and $K \leq G$, then $G \simeq K \times (G/K)$.

(f) The group $\mathbb{Z}/100\mathbb{Z}$ has a quotient isomorphic to $\mathbb{Z}/10\mathbb{Z}$.

(g) If G is an abelian group of order n and p is a prime number which divides n, then G has a quotient isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

(h) For any positive integers m and n, there are mn homomorphisms $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$.

(i) There exists a nontrivial homomorphism $\mathbb{Q} \to \mathbb{Z}$.

9. Let m and n be relatively prime positive integers. Show that we have an isomorphism

$$\phi : U(mn) \to U(m) \times U(n)$$

given by $\phi([a]_{mn}) = ([a]_m, [a]_n)$.

10. Let n be a positive integer and $d \mid n$. Show that the map $U(n) \to U(d)$ which sends $[a]_n \to [a]_d$ is surjective. (Note that we are not assuming d and n/d are relatively prime.)