1. [8, each part 2 marks] Consider the element $\sigma = (123)(246)(125)(1345)$ of S_6.

(a) Express σ as a product of disjoint cycles. No explanation necessary.

$\sigma = (254)(36)$

(b) Find $|\sigma|$.

$|\sigma| = \text{lcm}(2,3) = 6$

lengths of cycles in cycle decomposition of σ

(c) Express σ as a product of transpositions. No explanation necessary.

$\sigma = (25)(54)(36)$

(d) Express σ^{-1} as a product of disjoint cycles. No explanation necessary.

$\sigma^{-1} = (254)^{-1}(36)^{-1} = (245)(36)$.

2. [10, each part 2.5 marks] Determine if each statement below is true or false. Briefly justify your answers.

(a) Every two groups of order 17 are isomorphic. True

17 is prime, so every group of order 17 is cyclic. cyclic groups of the same order are isomorphic.

(b) Every finite subgroup of $\mathbb{C}^\times$ is cyclic. True

The only finite subgroups of $\mathbb{C}^\times$ are the μ_n and they are cyclic.

(c) The group $\mathbb{Z}/12$ has six subgroups. True

$\mathbb{Z}/12$ is cyclic of order 12, so for each divisor $d \mid 12$ it has a unique subgroup of order d.

12 has six divisors : $1, 2, 3, 4, 6, 12$

(d) If G is a group of order 10 with a unique subgroup of order 2 and a unique subgroup of order 5, then $G \simeq \mathbb{Z}/10$. True

Recall that if a group G of order n has a unique subgroup of order d for every divisor $d \mid n$, then G is cyclic.

3. [6] Let G be a group. Suppose that the map $\phi : G \to G$ given by $\phi(g) = g^{-1}$ is a homomorphism. Show that G is abelian.

Let $g, h \in G$. Then

$$\phi(g^{-1}h^{-1}) \underset{\substack{\downarrow \\ \text{b/c } \phi \\ \text{is a hom.}}}{=} \phi(g^{-1})\phi(h^{-1}) = (g^{-1})^{-1}(h^{-1})^{-1} = gh \quad (1)$$

On the other hand,

$$\phi(g^{-1}h^{-1}) = (g^{-1}h^{-1})^{-1} = hg. \quad (2)$$

(1), (2) $\Rightarrow$ $gh = hg$.

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4. [8] Let G and H be finite groups. Suppose there exists a surjective homomorphism $\phi : G \to H$. Show that if H has an element of order n, then G also has an element of order n.

3 marks Let $h \in H$ be an element of order n.

Since ϕ is surjective, there is $g \in G$ s.t. $\phi(g) = h$

1 Since G is finite, we have $|g| < \infty$.

2 Thus $|\phi(g)| \mid |g|$, i.e. $n \mid |g|$.

2 Let $|g| = n\ell$.

Then

$$|g^{\ell}| = \frac{|g|}{\gcd(|g|, \ell)} = \frac{n\ell}{\gcd(n\ell, \ell)} = \frac{n\ell}{\ell} = n.$$

5. [8] Let T be the subgroup of $GL_2(\mathbb{R})$ consisting of all the *diagonal* matrices in $GL_2(\mathbb{R})$. Show that given any group H, there is no homomorphism

$$GL_2(\mathbb{R}) \to H$$

whose kernel is equal to T.

4 marks

We know the kernel of any homomorphism is a normal subgroup of the domain. Thus to show there is no homomorphism $GL_2(\mathbb{R}) \to H$ with kernel T it is enough to show T is not normal in $GL_2(\mathbb{R})$.

4

Take $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in T$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

Then

$$BAB^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} * & -2 \\ * & * \end{pmatrix} \notin T.$$

Thus T is not a normal subgroup of $GL_2(\mathbb{R})$.

6. [12, each part 3 marks] Determine whether the two groups given in each part are isomorphic. Justify your answers.

(a) D_4 and S_3

Not isomorphic, because $|D_4| = 8$ and $|S_3| = 6$.

(b) $U(8)$ and μ_4

Not isomorphic, because $U(8)$ is not cyclic while μ_4 is.

(c) $\mathbb{Z}/10$ and μ_{10}

Isomorphic, because $\mathbb{Z}/10$ and μ_{10} are both cyclic of order 10.

(d) S_4 and D_{12}

Not isomorphic, because D_{12} has an element of order 12 (e.g. rotation by $\frac{2\pi}{12}$), but S_4 does not.

(Possible cycle types

4 ← Elements of this type have order 4
3, 1 ← " 3
2, 2 ← " 2
2, 1, 1 ← " 2
1, 1, 1, 1 ← " 1)

7. [8] Find the number of homomorphisms $\mu_4 \to A_7$. Explain.

4 marks

Since μ_4 is cyclic generated by $\zeta := e^{2\pi i/4}$, any homomorphism $\phi : \mu_4 \to A_7$ is determined by $\phi(\zeta)$. Moreover, $\phi(\zeta)$ can take value $\sigma \in A_7$ if and only if $\sigma^4 = e$.

Indeed, if $\phi(\zeta) = \sigma$, then $\sigma^4 = \phi(\zeta^4) = \phi(1) = e$, and conversely given arbitrary $\sigma \in A_7$ s.t. $\sigma^4 = e$, then the map $\zeta^n \xmapsto{\phi} \sigma^n$ is well-defined and is a homomorphism that sends $\zeta \mapsto \sigma$. (Argument for why this is well-defined: if $\zeta^n = \zeta^m$, then $4 \mid n-m$, which combined with $\sigma^4 = e$ gives $\sigma^n = \sigma^m$.)

Thus

$$|\mathrm{Hom}(\mu_4, A_7)| = |\{\sigma \in A_7 : \sigma^4 = e\}|.$$
$$= |\{\sigma \in A_7 : |\sigma| \in \{1,2,4\}\}|.$$

Here to the end 4 marks

Cycle types in S_7 that give elements of order 4 are 4,2,1 and 4,1,1,1. Elements of the former type are in A_7 while those of the latter type are not. Cycle types in S_7 giving elements of order 2 are 2,2,2,1, type 2,2,1,1,1, and type 2,1,1,1,1,1

Out of these only type 2,2,1,1,1 gives even permutations

Extra space. What you write here will not be graded unless you write next to the relevant question(s) "Continued on page 9".

$\Rightarrow$

$|\mathrm{Hom}(\mu_4, A_7)|$ = number of elements of type 4,2,1 in S_7

+ " 2,2,1,1,1

+ 1 (for identity element)

$$= \binom{7}{4} 3! \binom{3}{2} + \binom{7}{2}\binom{5}{2} \cdot \frac{1}{2} + 1$$

$$= \frac{7!}{4!3!} \cdot 3! \cdot 3 + 21 \cdot 10 \cdot \frac{1}{2} + 1$$

$$= 630 + 105 + 1$$

$$= \underline{736}.$$