

term-test-2-14532

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- 1. [8, each part 2 marks] Consider the element  $\sigma = (123)(246)(125)(1345)$  of  $S_6$ .
  - (a) Express  $\sigma$  as a product of disjoint cycles. No explanation necessary.

(b) Find |σ|.

$$|\sigma| = |cm(2,3) = 6$$
  
 $|ensths| = 6$  cycles in cycle decomposition of  $\sigma$ 

(c) Express  $\sigma$  as a product of transpositions. No explanation necessary.

(d) Express  $\sigma^{-1}$  as a product of disjoint cycles. No explanation necessary.

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- 2. [10, each part 2.5 marks] Determine if each statement below is true or false. Briefly justify your answers.
  - (a) Every two groups of order 17 are isomorphic.

17 is prime, so every group of order 17 is cyclic. cyclic groups of the same order are isomorphic

(b) Every finite subgroup of  $\mathbb{C}^{\times}$  is cyclic.

The only finite subgroups of Px are the Mn cyclic.

(c) The group  $\mathbb{Z}/12$  has six subgroups.

True

is cyclic of order 12, so for each divisor d 12

it has a unique subgroup of order d.

12 has six devisors: 1,2,3,4,6,12

(d) If G is a group of order 10 with a unique subgroup of order 2 and a unique subgroup of order 5, True then  $G \simeq \mathbb{Z}/10$ .

Recall onat it a group G of order n has a unique subgroup of order of for every divisor of In, ohen Gis ayclic.

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3. [6] Let G be a group. Suppose that the map  $\phi: G \to G$  given by  $\phi(g) = g^{-1}$  is a homomorphism. Show that G is abelian.

let g, h & G. Then

On the other hand,

$$\Phi(g^{-1}h^{-1}) = (g^{-1}h^{-1})^{-1} = hg$$



**4.** [8] Let G and H be finite groups. Suppose there exists a surjective homomorphism  $\phi: G \to H$ . Show that if H has an element of order n, then G also has an element of order n.

The let help be an element of order n.

Since & is sujective, where is  $g \in G$  s.t.  $\varphi(g) = h$ .

I (Since G is finite, we have  $|g| < \infty$ .

2 (Ohus  $|\varphi(g)| |g|$ , i.e. n |g|.

2 (let 
$$|g| = nl$$
.

 $|g^{\ell}| = \frac{|g|}{|g_{cd}(|g|, \ell)} = \frac{nl}{|g_{cd}(|nl, \ell)|} = \frac{nl}{|l|} = n$ .

5. [8] Let T be the subgroup of  $GL_2(\mathbb{R})$  consisting of all the *diagonal* matrices in  $GL_2(\mathbb{R})$ . Show that given any group H, there is no homomorphism

$$GL_2(\mathbb{R}) \to H$$

whose kernel is equal to T.

We know the kernel of any homomorphism is a normal subgroup of the domain. Thus to show there is no homomorphism  $GL_2(IR) \rightarrow H$  with kernel T. it is enough to show T is not normal in  $GL_2(IR)$ 

Take  $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in T$  and  $B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ .

The

 $BAB^{-1} = \binom{1}{12}\binom{1}{0-1}\binom{2}{-1}$ 

$$= \binom{1}{1} \binom{2}{2} \binom{2}{1} \binom{2}{1} \binom{2}{1}$$

$$= \begin{pmatrix} \times & -2 \\ \times & \times \end{pmatrix} \notin \top$$

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Thus T is not a normal subgroup of GLz(IR).

maks

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6. [12, each part 3 marks] Determine whether the two groups given in each part are isomorphic. Justify your answers.

(a) D<sub>4</sub> and S<sub>3</sub>

(b) U(8) and μ<sub>4</sub>

Not isomorphic, because U(8) is not cyclic while 14 is.

(c)  $\mathbb{Z}/10$  and  $\mu_{10}$ 

Isomorphic, because 72/10 and 110 are both cyclic of order 10.

(d) S<sub>4</sub> and D<sub>12</sub>

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Not isomorphic, because D<sub>12</sub> has an element of order 12 (e.g. rotation by 2th), but S4 does not.

(Possible cycle types

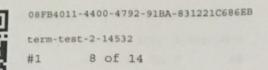
4 

Elements of This type have order 4

3,1 

2,2 

11



7. [8] Find the number of homomorphisms  $\mu_4 \to A_7.$  Explain. Since 14 is cyclic generated by 5:= e 211/4, any homomorphism Ø: M4 -> A7 is determined by Ø(3). Moreover, \$(5) can take value  $\sigma \in A_7$  if and only if  $\sigma^4 = e$ . Indeed, if  $\varphi(\xi) = \sigma$ , shen  $\sigma' = \varphi(\xi') = \varphi(1) = e$ , and convenely given arbidrary of Aq s.t. 04-e, then the map 5" \$ on is well-defined and is a homomorphism must sends & Ho. (Argument for why this is well-defined: if s= sm, shen 4/n-m, which combined with 04=e gives on = om.) |Hom(My, A7) | = | { \sigma \in A7 : \sigma^4 = e } |.

= \left\{0 \in A\_7 : |ole \{1,2,4\}\}\right\}.

Hore Cycle types in S\_7 onat give elements of order 4

to the end are 4,2,1 and 4,1,1,1. Elements of one former

type are in A\_7 while shore of the latter type are

not. Cycle types in S\_7 giving elements of order 2

are 2,2,2,1, type 8 of 14 2,2,1,1,1, and type 2,1,1,1,1,1

Out of these only type 2,2,1,1,1 gives even femorations.

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Extra space. What you write here will not be graded unless you write next to the relevant question(s) "Continued on page 9".