MAT247 Algebra II

Assignment 1

Solutions

1. (a) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Find the characteristic polynomial, eigenvalues, and a basis for each (nonzero) eigenspace of A. (Here take $F = \mathbb{R}$.)

(b) Let $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ be the map given by $L_A(x) = Ax$. Give a basis β of $\mathbb{R}^2$ such that $[L_A]_\beta$ is diagonal. No explanation is necessary.

(c) Let $x = (a\ b)^t$ (where t denotes the transpose). Find a formula for $A^n x$.

(d) Let (a_n) be the Fibonacci sequence, defined by $a_1 = 1$, $a_2 = 2$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$. Find a non-recursive formula for a_n. (Suggestion: For $n \geq 1$, set $x_n = (a_n\ a_{n+1})^t$. Then $x_n = Ax_{n-1}$.)

Solution: (a) One easily calculates $p_A(t) = \det(A - tI) = t^2 - t - 1$. It has two real roots, and hence two eigenvalues, $\lambda_\pm = (1 \pm \sqrt{5})/2$ (this means $\lambda_+ = (1 + \sqrt{5})/2$ and $\lambda_- = (1 - \sqrt{5})/2$). Denote the eigenspaces of $\lambda_\pm$ respectively by $E_\pm$. Then $\{v_\pm\}$ with $v_\pm = \begin{pmatrix} 1 \\ \lambda_\pm \end{pmatrix}$ is a basis for

$$E_\pm = N \begin{pmatrix} -\lambda_\pm & 1 \\ 1 & 1 - \lambda_\pm \end{pmatrix}.$$

(b) Let $\beta = \{v_+, v_-\}$. Then $[L_A]_\beta = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$.

(c) Let $P = (v_+\ v_-)$ (i.e. the first column of P is v_+ and its second column is v_-). Let γ be the standard basis of $\mathbb{R}^2$. Then $[I]_\beta^\gamma = P$ (with β as in (b) and I the identity map on $\mathbb{R}^2$), and by the change of basis formula

$$A = [L_A]_\gamma = [I]_\beta^\gamma [L_A]_\beta [I]_\gamma^\beta = P[L_A]_\beta P^{-1},$$

so that for any integer n,

$$A^n = (P[L_A]_\beta P^{-1})^n \overset{\text{why}}{=} P([L_A]_\beta)^n P^{-1} = \frac{1}{\lambda_- - \lambda_+} \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix} \begin{pmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{pmatrix} \begin{pmatrix} \lambda_- & -1 \\ -\lambda_+ & 1 \end{pmatrix}.$$

We leave it to the reader to simplify this and write $A^n(a\ b)^t$.

(d) Let x_n be as in the suggestion. Then from the definition of the sequence (a_n) one sees that $x_n = Ax_{n-1}$ for each n. Using this successively we see that

$$x_n = Ax_{n-1} = A^2 x_{n-2} = \cdots = A^{n-1} x_1.$$

Substituting A^{n-1} from part (c) and $x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ after simplification we get

$$x_n = \begin{pmatrix} \frac{\lambda_+^{n-1}(\lambda_- - 2) - \lambda_-^{n-1}(\lambda_+ - 2)}{\lambda_- - \lambda_+} \\ * \end{pmatrix},$$

so that (comparing the first entries)

$$a_n = \frac{\lambda_+^{n-1}(\lambda_- - 2) - \lambda_-^{n-1}(\lambda_+ - 2)}{\lambda_- - \lambda_+}.$$

REMARK. (1) The expression above can be simplified to

$$a_n = \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\sqrt{5}}. \tag{1}$$

To get this nicer expression directly, add the two terms $a_{-1} = 0$ and $a_0 = 1$ to the sequence. Then setting $x_n = (a_n\ a_{n+1})^t$ for $n \geq -1$, we have $x_n = A^{n+1}x_{-1}$. The above procedure will directly result in Eq. (1).

(2) Using the formula we found for a_n, since $|\lambda_-| < 1$, we easily see $\frac{a_{n+1}}{a_n} \to \lambda_+ = (1+\sqrt{5})/2$. The number $(1+\sqrt{5})/2$ is called the *golden ratio*, which has a long and rich history. You should read about it on Wikipedia.

2. Let V be a finite-dimensional vector space over a field F of characteristic $\neq 2$. Let $T : V \to V$ be a linear operator satisfying $T^2 = I$ (where T^2 means the composition $T \circ T$ and I is the identity map on V). Show that T is diagonalizable. (Suggestion: Show that $V = E_1 \oplus E_{-1}$, where $E_\lambda = \ker(T - \lambda I)$ is the eigenspace for λ.)

Solution: Following the suggestion we will show that we have a decomposition $V = E_1 \oplus E_{-1}$. This will prove the result, as if $\beta_\pm$ is a basis of $E_{\pm 1}$, then $\beta_+ \cup \beta_+$ will be a basis of V which consists of eigenvectors of T.

To show that $V = E_1 \oplus E_{-1}$, we need to show that (i) $V = E_1 + E_{-1}$, and (ii) $E_1 \cap E_{-1} = 0$. For (i), given $v \in V$, since $\mathrm{char}(F) \neq 2$, we can write v as $v = v_+ + v_-$, where

$$v_+ = \frac{v + T(v)}{2},\ v_- = \frac{v - T(v)}{2}.$$

Using the fact that $T^2 = I$ one easily check that $v_+ \in E_1$ and $v_- \in E_{-1}$. For (ii), let $v \in E_1 \cap E_{-1}$. Then $T(v) = v$ (as $v \in E_1$) and $T(v) = -v$ (as $v \in E_{-1}$). Thus $v = -v$, or in other words $2v = 0$. Since $2 \neq 0$ is our field, this implies $v = 0$.

3. Let F be a field and $A \in M_{n\times n}(F)$. Show that A is diagonalizable over F (which by definition, means that the map $L_A : F^n \to F^n$ given by $v \to Av$ is diagonalizable) if and only if there exists a matrix $Q \in M_{n\times n}(F)$ such that $Q^{-1}AQ$ is diagonal.

Solution: Throughout the solution $L_A : F^n \to F^n$ is the map left multiplication by A.

Suppose A is diagonalizable over F. Then there exists a basis $\beta = \{v_1, \ldots, v_n\}$ of F^n the elements of which are eigenvectors of A. Then the matrix $[L_A]_\beta$ is diagonal (why?). Let $Q \in M_{n\times n}(F)$ be the matrix whose j-th column is v_j. Then the change of basis formula implies $Q^{-1}AQ = [L_A]_\beta$ (why?).

Conversely, suppose there exists a matrix $Q \in M_{n\times n}(F)$ such that $Q^{-1}AQ$ is diagonal. Let v_j be the j-th column of Q. Then $\beta = \{v_1, \ldots, v_n\}$ is a basis of F^n (why?). By the change of basis formula $[L_A]_\beta = Q^{-1}AQ$. In particular, $[L_A]_\beta$ is diagonal, hence L_A is diagonalizable, i.e. A is diagonalizable over F.

4. (a) Let V be a vector space over $\mathbb{C}$. Then V can also be considered as a vector space over $\mathbb{R}$. Show that if $\{v_1, \ldots, v_n\}$ is a basis of V over $\mathbb{C}$, then $\{v_1, \ldots, v_n, iv_1, \ldots, iv_n\}$ is a basis of V over $\mathbb{R}$. (In particular, if V has dimension n as a complex vector space, then it has dimension $2n$ as a real vector space.)

(b) Let V be an n-dimensional vector space over $\mathbb{C}$ and $T : V \to V$ a linear operator. Let $f(t)$ be the characteristic polynomial of T; thus $f(t)$ is a polynomial of degree n with coefficients in

$\mathbb{C}$. Let $g(t)$ be the characteristic polynomial of T, considered as a linear operator on the underlying real vector space. Thus $g(t)$ is a polynomial of degree $2n$ with real coefficients (by (a)). Prove that $g(t) = f(t)\bar{f}(t)$, where bar denotes complex conjugation. (The complex conjugate of a polynomial is the polynomial obtained by taking the complex conjugates of the coefficients. That is, if $f(t) = \sum a_r t^r$, then $\bar{f}(t) := \sum \overline{a_r} t^r$.)

Solution: (a) Let $\gamma = \{v_1, \ldots, v_n, iv_1, \ldots, iv_n\}$. We first show that γ spans V over $\mathbb{R}$. Let $v \in V$. Since β spans V over $\mathbb{C}$, there exist $a_1, \ldots, a_n \in \mathbb{C}$ such that $v = \sum_j a_j v_j$. Writing $a_j = b_j + ic_j$ with $b_j, c_j \in \mathbb{R}$, we have $v = \sum_j b_j v_j + \sum_j c_j(iv_j)$, so that v is in the span of γ.

Now we show γ is linearly independent over $\mathbb{R}$. Suppose b_j, c_j $(1 \le j \le n)$ are real numbers such that $\sum_j b_j v_j + \sum_j c_j(iv_j) = 0$. Then this can be rewritten as $\sum_j (b_j + ic_j)v_j = 0$. The linear independence of the v_j over $\mathbb{C}$ implies that $b_j + ic_j = 0$ for all j, which in turn implies that $b_j = c_j = 0$ for all j.

(b) Let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V over $\mathbb{C}$. Denote the matrix of T with respect to β by A; it is an element of $M_{n\times n}(\mathbb{C})$. Write $A = B + iC$ with $B, C \in M_{n\times n}(\mathbb{R})$. (Denoting the $k\ell$ entry of a matrix M by $M_{k\ell}$ we have $A_{k\ell} = B_{k\ell} + iC_{k\ell}$.)

Let $\gamma = \{v_1, \ldots, v_n, iv_1, \ldots, iv_n\}$. By (a), γ is a basis of V over $\mathbb{R}$. We find the matrix of T (considered as a real linear transformation) with respect to γ. For $\ell \le n$, we have

$$T(v_\ell) = \sum_k A_{k\ell} v_k = \sum_k B_{k\ell} v_k \; + \; \sum_k C_{k\ell}(iv_k)$$

and

$$T(iv_\ell) \overset{\mathbb{C}\text{-linearity of } T}{=} iT(v_\ell) = \sum_k iA_{k\ell} v_k = \sum_k (-C_{k\ell}) v_k \; + \; \sum_k B_{k\ell}(iv_k).$$

Thus

$$[T]_\gamma = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

(this is an element of $M_{2n\times 2n}(\mathbb{R})$). The characteristic polynomial of T as a real operator is thus

$$g(t) = \det([T]_\gamma - tI) = \det \begin{pmatrix} B - tI & -C \\ C & B - tI \end{pmatrix}.$$

(With abuse of notation we are using the same notation for the $n \times n$ and $2n \times 2n$ identity matrices, but this should not lead to any confusion.) The matrix $[T]_\gamma - tI$ is a matrix with entries in the polynomial ring $\mathbb{R}[t]$ (or the function field $\mathbb{R}(t)$). We can "extend the scalars" and think of $[T]_\gamma - tI$ as a matrix with entries in $\mathbb{C}[t]$ (or if you prefer, $\mathbb{C}(t)$). The determinant $\det([T]_\gamma - tI)$ is the same no matter if the matrix $[T]_\gamma - tI$ is regarded as a matrix with entries in $\mathbb{R}[t]$ or $\mathbb{C}[t]$. Recall that adding a scalar multiple of a row (resp. column) to another row (resp. column) does not change the determinant. By our previous observation, we may use scalars in $\mathbb{C}$ for this (in fact, $\mathbb{C}(t)$ if we wish). Adding i times rows $n+1, \ldots, 2n$ respectively to rows $1, \ldots, n$ and then adding $-i$ times columns $1, \ldots, n$ to columns $n+1, \ldots, 2n$, we have

$$g(t) = \det \begin{pmatrix} B - tI & -C \\ C & B - tI \end{pmatrix} = \det \begin{pmatrix} B + iC - tI & -C + Bi - tiI \\ C & B - tI \end{pmatrix} = \det \begin{pmatrix} B + iC - tI & 0 \\ C & B - Ci - tI \end{pmatrix}.$$

Denoting by $\overline{A}$ the matrix obtained by taking the complex conjugates of the entries of A (i.e. $\overline{A} = B - Ci$)), we thus have

$$g(t) \overset{\text{why}}{=} \det(B + iC - tI)\det(B - Ci - tI) = \det(A - tI)\det(\overline{A} - tI) \overset{\text{See the remark below}}{=} f(t)\bar{f}(t),$$

as desired.

REMARK. Let $A \in M_{n\times n}(\mathbb{C})$. Then we have the following relation between the characteristic polynomials $p_A(t)$ and $p_{\overline{A}}(t)$ of A and $\overline{A}$:

$$p_{\overline{A}}(t) = \overline{p_A}(t)$$

(where as in the problem the complex conjugate of a polynomial is obtained by taking complex conjugates of the coefficients). Here is one way to see this: Let $\rho : \mathbb{C}[t] \to \mathbb{C}[t]$ be the map given by $\rho(f(t)) = \overline{f}(t)$. Then a straightforward computation using the facts $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{zz}$ for $z, w \in \mathbb{C}$ shows that $\rho(f(t)+g(t)) = \rho(f(t))+\rho(g(t))$ and $\rho(f(t)g(t)) = \rho(f(t))\rho(g(t))$ (that is, ρ is a *ring homomorphism*). For simplicity of notation, let $B = A - tI$. Then

$$f_A(t) = \sum_{\sigma\in S_n} sgn(\sigma) B_{1\sigma(1)} \dots B_{n\sigma(n)}.$$

Using the fact that ρ is a ring homomorphism, we get

$$\overline{f_A}(t) = \rho(f_A(t)) = \sum_{\sigma\in S_n} sgn(\sigma)\rho(B_{1\sigma(1)}) \dots \rho(B_{n\sigma(n)}).$$

The expression on the right is just $p_{\overline{A}}(t)$.

5. Let F be a field and V a finite-dimensional vector space over F. Let $V^\vee$ denote the dual space of V (i.e. $V^\vee$ is the set of all linear maps $V \to F$, with addition and scalar multiplication defined as follows: given $f, g \in V^\vee$ and $c \in F$, the maps $f + g : V \to F$ and $cf : V \to F$ are given by $(f + g)(v) = f(v) + g(v)$ and $(cf)(v) = c \cdot f(v)$). Let $T : V \to V$ be a linear operator. Then given any $f \in V^\vee$, being a composition of linear transformations, $f \circ T : V \to F$ is also linear. Let T^t (called the transpose or the dual of T) be the map $V^\vee \to V^\vee$ defined by $T^t(f) = f \circ T$. You can check that T^t is indeed linear (but you don't have to include the argument in your solution). Show that the characteristic polynomials of T and T^t are equal. (Suggestion: Let β be a basis of V. Let γ be the basis of $V^\vee$ dual to β. Try to relate $[T]_\beta$ and $[T^t]_\gamma$.)

Solution: Let β be a basis of V and γ be the basis of $V^\vee$ dual to β. We will show that $[T^\vee]_\gamma = ([T]_\beta)^t$. This will prove the result, as then the two matrices $[T^\vee]_\gamma - tI$ and $[T]_\beta - tI$ (with entries in $F[t]$) are transposes of one another, and hence have the same determinant.

Let $\beta = \{v_1, \dots, v_n\}$. Then (by definition) $\gamma = \{{v_1}^\vee, \dots, {v_n}^\vee\}$, where ${v_i}^\vee : V \to F$ is the linear map satisfying

$${v_i}^\vee(v_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

Let $[T]_\beta = (A_{ij})$ and $[T^\vee]_\gamma = (B_{ij})$. The goal is to show $B_{ij} = A_{ji}$. We have

$$T^\vee({v_j}^\vee) = \sum_k B_{kj} {v_k}^\vee.$$

Evaluating both sides at v_i , in view of $T^\vee({v_j}^\vee) = {v_j}^\vee \circ T$ and the definition of the ${v_k}^\vee$, we get

$${v_j}^\vee(T(v_i)) = B_{ij}$$

(as ${v_k}^\vee(v_i) = 0$ for $k \neq i$). On the other hand, we have

$${v_j}^\vee(T(v_i)) = {v_j}^\vee(\sum_k A_{ki} v_k) = A_{ji}.$$