MAT247 Algebra II

Assignment 1

Due Friday Jan 18 at 11:59 pm
(to be submitted on Crowdmark)

Please write your solutions neatly and clearly. Note that due to time limitations, some questions may not be graded.

1. (a) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Find the characteristic polynomial, eigenvalues, and a basis for each (nonzero) eigenspace of A. (Here take $F = \mathbb{R}$.)

(b) Let $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ be the map given by $L_A(x) = Ax$. Give a basis β of $\mathbb{R}^2$ such that $[L_A]_\beta$ is diagonal. No explanation is necessary.

(c) Let $x = (a\ b)^t$ (where t denotes the transpose). Find a formula for $A^n x$.

(d) Let (a_n) be the Fibonacci sequence, defined by $a_1 = 1$, $a_2 = 2$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$. Find a non-recursive formula for a_n. (Suggestion: For $n \geq 1$, set $x_n = (a_n\ a_{n+1})^t$. Then $x_n = Ax_{n-1}$.)

2. Let V be a finite-dimensional vector space over a field F of characteristic $\neq 2$. Let $T : V \to V$ be a linear operator satisfying $T^2 = I$ (where T^2 means the composition $T \circ T$ and I is the identity map on V). Show that T is diagonalizable. (Suggestion: Show that $V = E_1 \oplus E_{-1}$, where $E_\lambda = \ker(T - \lambda I)$ is the eigenspace for λ.)

3. Let F be a field and $A \in M_{n\times n}(F)$. Show that A is diagonalizable over F (which by definition, means that the map $L_A : F^n \to F^n$ given by $v \to Av$ is diagonalizable) if and only if there exists a matrix $Q \in M_{n\times n}(F)$ such that $Q^{-1}AQ$ is diagonal.

4. (a) Let V be a vector space over $\mathbb{C}$. Then V can also be considered as a vector space over $\mathbb{R}$. Show that if $\{v_1, \ldots, v_n\}$ is a basis of V over $\mathbb{C}$, then $\{v_1, \ldots, v_n, iv_1, \ldots, iv_n\}$ is a basis of V over $\mathbb{R}$. (In particular, if V has dimension n as a complex vector space, then it has dimension $2n$ as a real vector space.)

(b) Let V be an n-dimensional vector space over $\mathbb{C}$ and $T : V \to V$ a linear operator. Let $f(t)$ be the characteristic polynomial of T; thus $f(t)$ is a polynomial of degree n with coefficients in $\mathbb{C}$. Let $g(t)$ be the characteristic polynomial of T, considered as a linear operator on the underlying real vector space. Thus $g(t)$ is a polynomial of degree $2n$ with real coefficients (by (a)). Prove that $g(t) = f(t)\bar{f}(t)$, where bar denotes complex conjugation. (The complex conjugate of a polynomial is the polynomial obtained by taking the complex conjugates of the coefficients. That is, if $f(t) = \sum a_r t^r$, then $\bar{f}(t) := \sum \overline{a_r} t^r$.)

5. Let F be a field and V a finite-dimensional vector space over F. Let $V^\vee$ denote the dual space of V (i.e. $V^\vee$ is the set of all linear maps $V \to F$, with addition and scalar multiplication defined as follows: given $f, g \in V^\vee$ and $c \in F$, the maps $f + g : V \to F$ and $cf : V \to F$ are given by $(f + g)(v) = f(v) + g(v)$ and $(cf)(v) = c \cdot f(v)$). Let $T : V \to V$ be a linear operator. Then given any $f \in V^\vee$, being a composition of linear transformations, $f \circ T : V \to F$ is also linear. Let T^t (called the transpose or the dual of T) be the map $V^\vee \to V^\vee$ defined by $T^t(f) = f \circ T$. You can check that T^t is indeed linear (but you don't have to include the argument in your solution). Show that the characteristic polynomials of T and T^t are equal. (Suggestion: Let β be a basis of V. Let γ be the basis of $V^\vee$ dual to β. Try to relate $[T]_\beta$ and $[T^t]_\gamma$.)

Practice Problems: The following problems are for your practice. They are not to be handed in for grading.

From the textbook: End of section 5.1 exercises, in particular problems # 1, 3, 4, 13, 14 (this you will need for problem # 5 of the assignment), 15, 19, 20, 21, 22, 23

1. Let V be a finite-dimensional vector space over a field F and $T : V \to V$ a linear operator satisfying $T^2 = T$ (such an operator is called a projection). Show that T is diagonalizable. (Hint: Show that $V = E_0 \oplus E_1$.)

2. Determine if the statements below are true or false. Throughout T is a linear operator on a vector space V (i.e. $T : V \to V$ is a linear map).

(a) Zero is an eigenvalue of T if and only if T is not injective.
(b) Zero is an eigenvalue of T if and only if T is not invertible.
(c) If V is finite-dimensional, then zero is an eigenvalue of T if and only if T is not invertible.

3. Let $a_1, a_2 \in \mathbb{R}$. For $n \geq 3$, set $a_n = \frac{5}{2}a_{n-1} - a_{n-2}$. Show that the sequence (a_n) converges if and only if $(a_1, a_2) \in span\{(2, 1)\}$.

4. Let $\theta \in \mathbb{R}$ and $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(a) Show that for $\theta \notin \{n\pi : n \in \mathbb{Z}\}$, the matrix A is not diagonalizable over $\mathbb{R}$. (In other words, show that for such θ the map $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L_A(x) = Ax$ is not diagonalizable.)
(b) Show that A is diagonalizable over $\mathbb{C}$. Find a basis of $\mathbb{C}^2$ consisting of eigenvectors of A. Find matrices $P, D \in M_2(\mathbb{C})$ with D diagonal such that $P^{-1}AP = D$.

5. We will use the following notation in this question: given a vector space V over a field F, by $\dim_F(V)$ we mean the dimension of V as a vector space over F.

Let F and K be fields and $F \subset K$ (that is, F is a subfield of K).

(a) Let $A \in M_{m\times n}(F)$ (and hence also $A \in M_{m\times n}(K)$). Denote by N (resp. N') the nullspace of A in F^n (resp. K^n). Is $\dim_F(N) = \dim_K(N')$?
(b) Let $A \in M_{n\times n}(F)$ and $\lambda \in F$. Let E_λ (resp. E'_λ) be the eigenspace of A in F^n (resp K^n) corresponding to λ. Is $\dim_F(E_\lambda) = \dim_K(E'_\lambda)$?