MAT247 Algebra II

Assignment 2

Solutions

1. By calculating the characteristic polynomial, eigenvalues and dimensions of the eigenspaces of each map or matrix below, determine if the given map or matrix is diagonalizable. If a map or matrix is diagonalizable, diagonalize it (that is, give a basis consisting of its eigenvectors). The field F over which you consider the problem is given in each part.

(a) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ over an arbitrary field F

(b) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ over an arbitrary field F

(c) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $T(ax^2 + bx + c) = cx^2 + ax + b$ (Here $F = \mathbb{R}$.)

(d) $T : P_2(\mathbb{C}) \to P_2(\mathbb{C})$ given by $T(ax^2 + bx + c) = cx^2 + ax + b$ (Here $F = \mathbb{C}$.)

Solution: (a) The characteristic polynomial is $(1-t)^2(2-t)$. The eigenvalues are 1 and 2, with multiplicities 2 and 1, respectively. We leave it to the reader to check that $\dim(E_1) = 1$, which is less than the multiplicity of eigenvalue 1. Thus the matrix is not diagonalizanble. (Note that E_2 must be 1-dimensional, as the dimension of each eigenspace is no greater than the multiplicity of the corresponding eigenvalue.)

(b) The characteristic polynomials is the same as in part (a), but this time we easily see that E_1 has dimension 2 with a basis $\{(1,0,0),(0,1,0)\}$, and E_2 is 1-dimensional with a basis $\{(1,0,1)\}$. The sum of the dimensions of the eigenspaces equals 3 ($= \dim F^3$), so the matrix is diagonalizable. The basis $\{(1,0,0),(0,1,0),(1,0,1)\}$ of F^3 consists of eigenvectors of A.

(c) We calculate the matrix representation of T with respect to the basis $\beta = \{1, x, x^2\}$ of $P_2(\mathbb{R})$.

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Thus

$$p_T(t) = \det \begin{pmatrix} -t & 1 & 0 \\ 0 & -t & 1 \\ 1 & 0 & -t \end{pmatrix} = -t^3 + 1 = -(t-1)(t^2+t+1).$$

The characteristic polynomial does not split over $\mathbb{R}$, hence the map is not diagonalizable.

(d) The characteristic polynomial is the same as in part (c). It has three distinct roots in $\mathbb{C}$, namely $1, \omega = (1+\sqrt{3})/2$ and $\omega^2 = (1-\sqrt{3})/2$. Thus the map is diagonalizable. Note that since the eigenvalues all have multiplicity 1, all eigenspaces are 1-dimensional. We leave it to the reader to check that x^2+x+1, $x^2+\omega^2 x+\omega$, and $x^2+\omega x+\omega^2$ respectively belong to E_1, E_ω, and E_{ω^2} (and form a basis for them). Thus $\{x^2+x+1, x^2+\omega^2 x+\omega, x^2+\omega x+\omega^2\}$ is a basis of $P_2(\mathbb{C})$ consisting of eigenvectors of T.

2. Let V be a vector space and k an integer ≥ 1. For each integer $1 \leq i \leq k$, let V_i be a subspace of V. Show that the following statements are equivalent:

(i) For every $1 \le i \le k$,
$$V_i \cap \sum_{\substack{1\le j\le k\\ j\ne i}} V_j \;=\; 0.$$

(ii) If $v_i \in V_i$ for $1 \le i \le k$ and $\sum_{i=1}^{k} v_i = 0$, then $v_i = 0$ for all $1 \le i \le k$.

(iii) If for each $1 \le i \le k$, the set β_i is a linearly independent subset of V_i , then the β_i are pairwise disjoint (i.e. $\beta_i \cap \beta_j = \emptyset$ whenever $i \ne j$) and $\bigcup_{i=1}^{k} \beta_i$ is linearly independent.

Note: I am aware that the proof of this can be found in your textbook, but you should do it yourself. The empty sum (which appears in (i) when $k = 1$) is by convention defined to be zero.

Solution:

(i)$\Rightarrow$ (ii) : Suppose $v_i \in V_j$ for $1 \le i \le k$ and $\sum_{i=1}^{k} v_i = 0$. Fix i. We have $v_i = -\sum_{j\ne i} v_j$, so that v_i belongs to $V_i \cap \sum_{j\ne i} V_j$. By (i), $v_i = 0$.

(ii)$\Rightarrow$ (iii) : Suppose a linear combination of some elements of the β_i is zero. More precisely, for each i, let $v_{i,j}$ $(1 \le j \le m_i)$ be in β_i, and for some some scalars a_{ij},
$$\sum_{i=1}^{k}\sum_{j=1}^{m_i} a_{ij}v_{ij} = 0.$$
(Here again, empty sum is interpreted as zero.) Being a linear combination of elements of V_i, the element $\sum_{j=1}^{m_i} a_{ij}v_{ij}$ is in V_i, so that by (ii) we must have $\sum_{j=1}^{m_i} a_{ij}v_{ij} = 0$ for each i. The linear independence of β_i now implies that $a_{ij} = 0$ for all j. Thus all the a_{ij} are zero. It follows that the β_i are pairwise disjoint and their union is linearly independent.

(iii)$\Rightarrow$ (i) : Fix i and suppose there exists a nonzero element $v_i \in V_i \cap \sum_{\substack{1\le j\le k\\ j\ne i}} V_j$. Then there exist elements $v_j \in V_j$ $(j \ne i)$ such that $v_i = \sum_{j\ne i} v_j$. For each $1 \le \ell \le k$, let
$$\beta_\ell = \begin{cases} \{v_\ell\} & \text{if } v_\ell \ne 0 \\ \emptyset & \text{otherwise.} \end{cases}$$
In particular, $\beta_i = \{v_i\}$. Each β_ℓ is linearly independence, and hence by (iii), the β_ℓ are pairwise disjoint and their union is linearly independence. But
$$v_i - \sum_{\substack{j\ne i\\ v_j\ne 0}} v_j = 0$$
gives a linear dependence between the elements of $\bigcup_{1\le\ell\le k} \beta_\ell$.

3. Let V be a vector space over a field F. Let $T : V \to V$ be a linear map. Let W be a subspace of V. We say W is T-*invariant* if for every $w \in W$, we have $T(w) \in W$. Let W be T-invariant. Then T restricts to a linear map $W \to W$, which we denote by T_W (given by $T_W(w) = T(w)$). Let V/W be the quotient of V by W (to recall its definition and some useful results see Exercise 31 on page 23, Exercise 35 on page 58, and Exercise 40 on page 79 of the textbook).

(a) Show that $\overline{T} : V/W \to V/W$ given by $\overline{T}(v + W) = T(v) + W$ is well-defined and linear. (Being well-defined means that the definition makes sense. The reason we have to check this is because given $v + W \in V/W$, the formula for $\overline{T}(v + W)$ makes use of the representative v of the coset $v + W$. We need to make sure the output $\overline{T}(v + W)$ does not change if we choose a different representative for $v + W$. More explicitly, we need to make sure that if $v + W = v' + W$ for some $v, v' \in V$, then $T(v) + W = T(v') + W$.)

(b) Let V be finite-dimensional and W a nonzero proper T-invariant subspace. Denote the characteristic polynomials of T, T_W, and $\overline{T}$ respectively by $f(t)$, $g(t)$, and $h(t)$. Show that $f(t) = g(t)h(t)$. (Suggestion: Exercise 35 on page 58 can be useful.)

Solution: For any $v \in V$, below we write $\overline{v}$ for the element $v + W$ of V/W.

(a) First, let us check well-definedness. Let $v, v' \in V$ and $\overline{v} = \overline{v'}$. We need to show that $\overline{T(v)} = \overline{T(v')}$. The former (resp. latter) equation is equivalent to $v - v' \in W$ (resp. $T(v) - T(v') \in W$). Since $v - v' \in W$ and W is T-invariant, $T(v - v') \in W$. Since T is linear, $T(v - v') = T(v) - T(v')$.

Now we check that $\overline{T}$ is linear. Given $v \in V$ and a scalar c,

$$\overline{T}(c\overline{v}) \overset{(*)}{=} \overline{T}(\overline{cv}) \overset{(**)}{=} \overline{T(cv)} \overset{(\dagger)}{=} \overline{cT(v)} \overset{(*)}{=} c\overline{T(v)} \overset{(**)}{=} c\overline{T}(\overline{v}),$$

where in $(*)$, $(**)$, and $(\dagger)$ we respectively used the definition of scalar multiplication in V/W, the definition of the map $\overline{T}$, and linearity of T.

Given $v_1, v_2 \in V$, we have

$$\overline{T}(\overline{v_1} + \overline{v_2}) = \overline{T}(\overline{v_1 + v_2}) = \overline{T(v_1 + v_2)} = \overline{T(v_1) + T(v_2)} = \overline{T(v_1)} + \overline{T(v_2)} = \overline{T}(\overline{v_1}) + \overline{T}(\overline{v_2}).$$

(Make sure you know why each equality holds.)

(b) Let $\alpha = \{v_1, \dots, v_k\}$ be a basis of W. We extend α to a basis $\beta = \{v_1, \dots, v_n\}$ of V. Then $\gamma = \{\overline{v_{k+1}}, \dots, \overline{v_n}\}$ is a basis of V/W. Let $A = [T]_\beta$, with entries denoted by A_{ij}. Since W is T-invariant, for $1 \le j \le k$, we have

$$T_W(v_j) = T(v_j) = \sum_{i=1}^{k} A_{ij} v_i.$$

On the other hand, for $k + 1 \le j \le n$,

$$\overline{T}(\overline{v_j}) = \overline{T(v_j)} = \overline{\sum_{i=1}^{n} A_{ij} v_i} = \sum_{i=1}^{n} A_{ij} \overline{v_i} = \sum_{i=k+1}^{n} A_{ij} \overline{v_i}.$$

Thus the matrix $[T]_\beta$ (in block form) looks like

$$[T]_\beta = \begin{pmatrix} [T_W]_\alpha & * \\ 0 & [\overline{T}]_\gamma \end{pmatrix}.$$

Then $[T]_\beta - tI$ is block upper triangular with diagonal blocks $[T_W]_\alpha - tI$ and $[\overline{T}]_\gamma - tI$), and hence

$$f(t) = \det([T]_\beta - tI) = \det([T_W]_\alpha - tI)\det([\overline{T}]_\gamma - tI) = g(t)h(t).$$

4. (a) Let F be a field. Given a polynomial $f(t)$ of degree n over F we say $f(t)$ *splits* over F if it factors as

$$f(t) = c(t - \lambda_1)\cdots(t - \lambda_n)$$

for some $c \in F$ and (not necessarily distinct) $\lambda_1, \dots, \lambda_n \in F$. Let V be an n-dimensional vector space over F and $T : V \to V$ be a linear map whose characteristic polynomial splits over F. Show that there exists a basis β of V such that the matrix $[T]_\beta$ is upper triangular. (Suggestion: Argue by induction on $\dim(V)$. The previous problem can be useful.)

(b) Let $A = \begin{pmatrix} 2 & 0 & -1 \\ -1 & -1 & 1 \\ -1 & -4 & 2 \end{pmatrix} \in M_3(\mathbb{Q})$. Find a matrix $P \in M_3(\mathbb{Q})$ such that $P^{-1}AP$ is upper triangular, if such P exists. (Suggestion: Your proof for (a) tells you how to find such P.)

Solution: (a) We argue by induction on the dimension of V. If V is 1-dimensional, then the result is clear. Suppose $n \geq 2$ and the result holds for linear operators on vector spaces of dimension $n-1$. Let T be an operator on a vector space V of dimension n over F, such that the characteristic polynomial of T splits over F, say

$$p_T(t) = c(t-\lambda_1)\cdots(t-\lambda_n),$$

with the λ_i and c in F (comparing the leading coefficients we see $c = (-1)^n$). Being a root of $p_T(t)$ in F, λ_1 is an eigenvalue of T. Let w_1 be an eigenvector corresponding to λ_1. Let $W = span\{w_1\}$; it is a 1-dimensional T-invariant subspace of V. Thus V/W has dimension $n-1$. Let $\overline{T} : V/W \to V/W$ be the map induced by T on the quotient V/W (defined by $\overline{T}(\overline{v}) = \overline{T(v)}$ - note that here we used the fact that W is T-invariant). By Problem 3, we have

$$p_T(t) = p_{T_W}(t)p_{\overline{T}}(t) = (\lambda_1 - t)p_{\overline{T}}(t).$$

It follows that

$$p_{\overline{T}}(t) = (-1)^{n-1}(t-\lambda_2)\cdots(t-\lambda_n).$$

In particular, the characteristic polynomial of $\overline{T}$ splits over F. Since V/W has dimension $n-1$, by the induction hypothesis, there exists a basis γ of V/W such that $[\overline{T}]_\gamma$ is upper triangular. Suppose $\gamma = \{\overline{w_2}, \ldots, \overline{w_n}\}$, for some $w_2, \ldots, w_n \in V$. We claim that (1) $\beta = \{w_1, \ldots, w_n\}$ is a basis of V, and (2) $[T]_\beta$ is upper triangular. To see (1), since $\dim(V) = n$, it is enough to show that β is linearly independent. Suppose $\sum_{i=1}^{n} a_i w_i = 0$. Then, by linearity of the (quotient) map $V \to V/W$ sending $v \mapsto \overline{v}$ (or by the definition of the operations in the quotient vector space), and in view of $\overline{w_1} = 0$ (the zero of the quotient space), we have

$$\sum_{i=2}^{n} a_i \overline{w_i} = 0.$$

Linear independence of γ now implies that $a_i = 0$ for $2 \leq i \leq n$. Substituting back in $\sum_{i=1}^{n} a_i w_i = 0$, and on recalling $w_1 \neq 0$, we see that $a_1 = 0$ as well. Thus our first claim is established. Now (2) follows from that $[T]_\beta$ is of the form $\begin{pmatrix} \lambda_1 & * \\ 0 & [\overline{T}]_\gamma \end{pmatrix}$ (see the argument for 3b).

(b) Let $L_A : \mathbb{Q}^3 \to \mathbb{Q}^3$ be the map defined by $L_A(x) = Ax$. By the change of basis formula, if $P \in M_3(\mathbb{Q})$ is an invertible matrix with columns v_1, v_2, v_3 (where v_j is the j-th column), then the matrix $[L_A]_\beta$ of L_A with respect to the basis $\beta = \{v_1, v_2, v_3\}$ is equal to $P^{-1}AP$. Thus to answer the question, we shall find a basis β of $\mathbb{Q}^3$ such that $[L_A]_\beta$ is upper triangular. For this, we follow the process of the argument given in part (a).

A straightforward calculation shows that $p_{L_A}(t) = p_A(t) = -(t-1)^3$. Since the characteristic polynomial splits over $\mathbb{Q}$, a basis β as desired exists. One easily sees that $w_1 = (1\ 0\ 1)^t$ is an eigenvector (and that $\dim(E_1) = 1$). Set $W = span\{w_1\}$. Let $\overline{L_A} : \mathbb{Q}^3/W \to \mathbb{Q}^3/W$ be the map induced by L_A on the quotient (i.e. the map defined by $\overline{L_A}(\overline{v}) = \overline{L_A(v)}$). By 3(b), the characteristic polynomial of $\overline{L_A}$ is then $(t-1)^2$. If $\gamma = \{\overline{w_2}, \overline{w_3}\}$ is a basis of $\mathbb{Q}^3/W$ such that $[\overline{L_A}]_\gamma$ is upper triangular, then as we argued in (a), the matrix $[L_A]_\beta$ with $\beta = \{w_1, w_2, w_3\}$ is upper triangular. Thus our goal now is to a basis γ that *upper triangularizes* $\overline{L_A}$. Note that any basis

$\gamma = \{\overline{w_2}, \overline{w_3}\}$ of $\mathbb{Q}^3/W$ such that $\overline{w_2}$ is an eigenvector of $\overline{L_A}$ (corresponding to eigenvalue 1, the only eigenvalue of $\overline{L_A}$) will upper triangularize $\overline{L_A}$, as then the matrix $[\overline{L_A}]_\gamma$ looks like $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$. Thus we just need to find an eigenvector $\overline{w_2}$ of $\overline{L_A}$. We consider the equation $\overline{L_A}\overline{v} = \overline{v}$; note that since 1 is an eigenvalue of $\overline{L_A}$, this equation must have nonzero solutions in $\mathbb{Q}^3/W$. The equation can be rewritten as $\overline{Av} = \overline{v}$, which is equivalent to $Av - v \in W$. Thus we are looking for a $v \in V$ such that $(A - I)v = cw_1$ for some $c \in \mathbb{Q}$, and moreover $v \notin W$ (so that $\overline{v} \neq 0$). The condition $v \notin W = E_1(A)$ is guaranteed if $c \neq 0$ (why?). Writing $v = (x_1\ x_2\ x_3)^t$, we have

$$\begin{pmatrix} x_1 - x_3 \\ -x_1 - 2x_2 + x_3 \\ -x_1 - 4x_2 + x_3 \end{pmatrix} = c \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Taking say $c = 1$, we see a solution is $v = (0\ -\frac{1}{2}\ -1)^t$. We thus take $w_2 = (0\ -\frac{1}{2}\ -1)^t$ and, say, $w_3 = (1\ 0\ 0)^t$ (or any other vector not in $span\{w_1, w_2\}$). Then the matrix of $\overline{L_A}$ with respect to $\{\overline{w_2}, \overline{w_3}\}$ is upper triangular, and so it the matrix of L_A with respect to $\beta = \{w_1, w_2, w_3\}$. In view of our first observation in the solution, we can thus take

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

5. Let V be a vector space over a field F. We say a linear map $T : V \to V$ is *nilpotent* if there exists a positive integer m such that $T^m = 0$ (that is, $T^m(v) = 0$ for all $v \in V$). For instance, the differentiation map $P_n(F) \to P_n(F)$ is nilpotent.

(a) Let V be finite-dimensional and $T : V \to V$ a linear map such that for every $v \in V$, there exists an integer $k \geq 1$ (possibly depending on v) such that $T^k(v) = 0$. Show that T is nilpotent.

(b) Let $\dim(V) = n$ and $T : V \to V$ be a nilpotent linear map. Show that if λ is an eigenvalue of T, then $\lambda = 0$. Conclude that if the characteristic polynomial $p_T(t)$ of T splits over F, then $p_T(t) = (-1)^n t^n$.

Remark: The extra hypothesis here that $p_T(t)$ splits over F is actually not necessary, as it is automatically satisfied for a nilpotent map. See the practice problems.

Solution: (a) Let $\{v_1, \ldots, v_n\}$ be a basis of V. For each $1 \leq i \leq n$, let k_i be a positive integer such that $T^{k_i}(v_i) = 0$. Let m be the maximum of $k_1, \ldots, k_n$. Then T^m vanishes at all the v_i, and hence $T^m = 0$.

(b) Let λ be an eigenvalue of T. Let v be a corresponding eigenvector. Suppose $T^m = 0$. Then $0 = T^m(v) = \lambda^m v$. Since $v \neq 0$, it follows that λ^m is zero, and hence so is λ.

Now suppose $p_T(t)$ splits over F. Thus we have

$$p_T(t) = (-1)^n \prod_{i=1}^{n} (t - \lambda_i)$$

for some $\lambda_1, \ldots, \lambda_n \in F$. Each λ_i is a root of the characteristic polynomial, and hence is an eigenvalue of T. By the first assertion, all the λ_i are zero, so that $p_T(t) = (-1)^n t^n$.