MAT247 Algebra II

Assignment 2

Due Friday Jan 25 at 11:59 pm
(to be submitted on Crowdmark)

Please write your solutions neatly and clearly. Note that due to time limitations, some questions may not be graded.

1. By calculating the characteristic polynomial, eigenvalues and dimensions of the eigenspaces of each map or matrix below, determine if the given map or matrix is diagonalizable. If a map or matrix is diagonalizable, diagonalize it (that is, give a basis consisting of its eigenvectors). The field F over which you consider the problem is given in each part.

(a) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ over an arbitrary field F

(b) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ over an arbitrary field F

(c) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $T(ax^2 + bx + c) = cx^2 + ax + b$ (Here $F = \mathbb{R}$.)

(d) $T : P_2(\mathbb{C}) \to P_2(\mathbb{C})$ given by $T(ax^2 + bx + c) = cx^2 + ax + b$ (Here $F = \mathbb{C}$.)

2. Let V be a vector space and k an integer ≥ 1. For each integer $1 \leq i \leq k$, let V_i be a subspace of V. Show that the following statements are equivalent:

(i) For every $1 \leq i \leq k$,

$$V_i \cap \sum_{\substack{1 \leq j \leq k \\ j \neq i}} V_j \;=\; 0.$$

(ii) If $v_i \in V_i$ for $1 \leq i \leq k$ and $\sum_{i=1}^{k} v_i = 0$, then $v_i = 0$ for all $1 \leq i \leq k$.

(iii) If for each $1 \leq i \leq k$, the set β_i is a linearly independent subset of V_i, then the β_i are pairwise disjoint (i.e. $\beta_i \cap \beta_j = \emptyset$ whenever $i \neq j$) and $\bigcup_{i=1}^{k} \beta_i$ is linearly independent.

Note: I am aware that the proof of this can be found in your textbook, but you should do it yourself. The empty sum (which appears in (i) when $k = 1$) is by convention defined to be zero.

3. Let V be a vector space over a field F. Let $T : V \to V$ be a linear map. Let W be a subspace of V. We say W is T-*invariant* if for every $w \in W$, we have $T(w) \in W$. Let W be T-invariant. Then T restricts to a linear map $W \to W$, which we denote by T_W (given by $T_W(w) = T(w)$). Let V/W be the quotient of V by W (to recall its definition and some useful results see Exercise 31 on page 23, Exercise 35 on page 58, and Exercise 40 on page 79 of the textbook).

(a) Show that $\overline{T} : V/W \to V/W$ given by $\overline{T}(v + W) = T(v) + W$ is well-defined and linear. (Being well-defined means that the definition makes sense. The reason we have to check this is because given $v + W \in V/W$, the formula for $\overline{T}(v + W)$ makes use of the representative v of the coset $v + W$. We need to make sure the output $\overline{T}(v + W)$ does not change if we choose a different representative for $v + W$. More explicitly, we need to make sure that if $v + W = v' + W$ for some $v, v' \in V$, then $T(v) + W = T(v') + W$.)

(b) Let V be finite-dimensional and W a nonzero proper T-invariant subspace. Denote the characteristic polynomials of T, T_W, and $\overline{T}$ respectively by $f(t)$, $g(t)$, and $h(t)$. Show that $f(t) = g(t)h(t)$. (Suggestion: Exercise 35 on page 58 can be useful.)

4. (a) Let F be a field. Given a polynomial $f(t)$ of degree n over F we say $f(t)$ *splits* over F if it factors as

$$f(t) = c(t - \lambda_1)\cdots(t - \lambda_n)$$

for some $c \in F$ and (not necessarily distinct) $\lambda_1, \dots, \lambda_n \in F$. Let V be an n-dimensional vector space over F and $T : V \to V$ be a linear map whose characteristic polynomial splits over F. Show that there exists a basis β of V such that the matrix $[T]_\beta$ is upper triangular. (Suggestion: Argue by induction on $\dim(V)$. The previous problem can be useful.)

(b) Let $A = \begin{pmatrix} 2 & 0 & -1 \\ -1 & -1 & 1 \\ -1 & -4 & 2 \end{pmatrix} \in M_3(\mathbb{Q})$. Find a matrix $P \in M_3(\mathbb{Q})$ such that $P^{-1}AP$ is upper triangular, if such P exists. (Suggestion: Your proof for (a) tells you how to find such P.)

5. Let V be a vector space over a field F. We say a linear map $T : V \to V$ is *nilpotent* if there exists a positive integer m such that $T^m = 0$ (that is, $T^m(v) = 0$ for all $v \in V$). For instance, the differentiation map $P_n(F) \to P_n(F)$ is nilpotent.

(a) Let V be finite-dimensional and $T : V \to V$ a linear map such that for every $v \in V$, there exists an integer $k \geq 1$ (possibly depending on v) such that $T^k(v) = 0$. Show that T is nilpotent.

(b) Let $\dim(V) = n$ and $T : V \to V$ be a nilpotent linear map. Show that if λ is an eigenvalue of T, then $\lambda = 0$. Conclude that if the characteristic polynomial $p_T(t)$ of T splits over F, then $p_T(t) = (-1)^n t^n$.

Remark: The extra hypothesis here that $p_T(t)$ splits over F is actually not necessary, as it is automatically satisfied for a nilpotent map. See the practice problems.

Practice Problems: The following problems are for your practice. They are not to be handed in for grading.

From the textbook: End of section 5.2 exercises except the ones that mention the word "multiplicity". In particular, do problems # 1 (ignore f for now), 2, 3, 4, 7, 8, 9, 12, 17-19, 22.

Extra problems:

1. Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V. Suppose there are T-invariant subspaces U and W of V such that $V = U \oplus W$. Denote the restriction of T to U and W respectively by T_U and T_W. Let the characteristic polynomials of T, T_U, and T_W be respectively $f(t)$, $g(t)$ and $h(t)$. Show that $f(t) = g(t)h(t)$.

2. Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V. Let $U \subset V$ be a T-invariant subspace. Is there necessarily a T-invariant subspace W such that $V = U \oplus W$? Either prove such W always exists or give a counter-example.

3. Let V be a vector space over $\mathbb{C}$. Define a new scalar product $\star$ on V as follows: given $z \in \mathbb{C}$ and $v \in V$, define $z \star v := \bar{z}v$.

(a) Show that V with its addition and $\star$ as scalar multiplication is again a vector space over $\mathbb{C}$.

(b) To distinguish between the two complex vector space structures on V, let us use V' to refer to V with scalar multiplication $\star$, and V for the vector space with the original scalar multiplication. Construct an isomorphism between V and V'. Is your isomorphism natural? (If we make some choices in the construction (such as choosing a basis) and the constructed map depends on those choices, we would say the map is not natural.)

(c) Let $T : V \to V$ be a linear map. Show that T considered as a function $V' \to V'$ is again linear.

(d) Let us use T' to refer to the map $T : V \to V$, if it is considered as a linear map $V' \to V'$. How are the characteristic polynomials (resp. determinants) of T and T' related to one another?

4. Let F be a field and $A, B \in M_n(F)$ be similar. Show that the eigenvalues of A and B are the same, and that moreover, for every $\lambda \in F$, the two eigenspaces $E_\lambda(A)$ and $E_\lambda(B)$ have the same dimension.

5. Suppose $T : V \to V$ is nilpotent and diagonalizable. Show that $T = 0$.

6. Let V be a nonzero finite-dimensional vector space over a field F, and $T : V \to V$ a nilpotent map. Let N be the smallest positive integer such that $T^N = 0$.

(a) Prove that for each $0 \le i < N$, $\ker(T^i) \subsetneq \ker(T^{i+1})$. (By convention, set $T^0 = I$. The notation $A \subsetneq$ means A is a proper subset of B, i.e. $A \subset B$ and $A \ne B$. Suggestion: That $\ker(T^i) \subset \ker(T^{i+1})$ is clear. To prove properness, use the rank-nullity theorem to deduce that $\ker(T^i) = \ker(T^{i+1})$ implies $\mathrm{Im}(T^i) = \mathrm{Im}(T^{i+1})$. Show that $\mathrm{Im}(T^i) = \mathrm{Im}(T^{i+1})$ implies that $\mathrm{Im}(T^i) = \mathrm{Im}(T^\ell)$ for every $\ell \ge i$.)

(b) Conclude that $N \le \dim(V)$.

7. The goal of this question is to find the characteristic polynomial of a nilpotent map, without assuming that the characteristic polynomial splits.

Let V be an n-dimensional vector space over a field F, and $T : V \to V$ a nilpotent map. Assume $V \ne 0$ (i.e. $n \ge 1$).

(a) Construct a basis β of V such that $[T]_\beta$ is upper triangular, with zeros on the diagonal. (Hint: Let N be the smallest positive integer such that $T^N = 0$. Consider

$$0 \subsetneq \ker(T) \subsetneq \ker(T^2) \subsetneq \cdots \subsetneq \ker(T^N) = V.$$

Take a basis of $\ker(T)$, extend to a basis of $\ker(T^2)$, etc. to finally get a basis of $\ker(T^N) = V$.)

(b) Deduce that $p_T(t) = (-1)^n t^n$. (In particular, you have proved that the characteristic polynomial of T splits over F.)

8. There is a quicker way of proving the result about the characteristic polynomial of a nilpotent map with no hypothesis on the splitting of the characteristic polynomial, but it requires a result from abstract algebra that is not part of our course, namely the following theorem:

THEOREM. For any field F there exists an algebraically closed field K which contains F.

Taking the theorem for granted, give another argument for the fact that the characteristic polynomial of a nilpotent operator on an n-dimensional vector space is $(-1)^n t^n$. (You want to somehow replace the field with an algebraically closed field, so that the splitting of the characteristic polynomial is guaranteed.)

9. Let $V_1, \ldots, V_k$ be subspaces of V. Show that the conditions of Problem 2 of the assignment are also equivalent to statement (iv) below, and if the V_i are finite-dimensional, moreover equivalent to statement (v) below:

(iv) Every element of $\sum_{i=1}^{k} V_i$ can be expressed uniquely as a sum $\sum_{i=1}^{k} v_i$ with $v_i \in V_i$ for each i.

(v) $\dim \sum_{i=1}^{k} V_i = \sum_{i=1}^{k} \dim V_i$.

(You can easily show that $(iv) \Leftrightarrow (ii)$, and when the V_i are finite-dimensional, $(v) \Leftrightarrow (iii)$.)