MAT247 Algebra II

Assignment 3

Due Friday Feb 1 at 11:59 pm
(to be submitted on Crowdmark)

Please write your solutions neatly and clearly. Note that due to time limitations, some questions may not be graded.

1. Let T be a linear operator on a finite-dimensional vector space over a field F. Suppose that the characteristic polynomial of T is $f(t) = \pm t^2(t-1)^3(t^p+1)$, where p is a prime number.

(a) Suppose $F = \mathbb{R}$. Can T be diagonalizable? What are the eigenvalues of T? What are the possible values for the dimension of each eigenspace of T? Give an example for each possibility. (Suggestion: Construct your examples using block diagonal matrices. Exercise 19 of 5.4 gives a way of constructing a matrix with a given characteristic polynomial.)

(b) Suppose $F = \mathbb{C}$. Can T be diagonalizable? If yes, suppose T is diagonalizable. What is the dimension of each eigenspace of T?

(c) Suppose F has characteristic p. What are the eigenvalues of T? What are the possible values of the dimension of each eigenspace of T? (Suggestion: Expand $(t+1)^p$.)

(d) Back to a general F of arbitrary characteristic, show that T is not surjective.

2. For $A \in M_{n\times n}(\mathbb{C})$, define

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

One can show that the sum converges for every A.

(a) Ignoring all convergence-related questions, show that if $AB = BA$, then $e^{A+B} = e^A e^B$.

(b) Show that if A is nilpotent, then the characteristic polynomial of e^A is $(-1)^n(t-1)^n$. (An element of $M_{n\times n}(F)$ with characteristic polynomial $(-1)^n(t-1)^n$ is called a *unipotent* matrix. Hint: How are the characteristic polynomials of B and $B + \lambda I$ related to one another?)

(c) Does it make sense to define e^A with the same formula as above for an $n \times n$ matrix A with entries in an arbitrary field of characteristic zero? What if A is nilpotent?

(d) Can we define e^A with the formula as above for a nilpotent matrix A with entries in a field of positive characteristic?

3. Let T be a linear operator on a nonzero finite-dimensional vector space V over a field F. The *minimal polynomial* of T is the monic† polynomial $f(t)$ of smallest degree such that $f(T) = 0$. It is easy to show that the minimal polynomial exists and is unique, and by the Cayley-Hamilton theorem, its degree is $\leq \dim(V)$. You don't have to include the argument for these facts in your solution.

Suppose $f(t)$ is the minimal polynomial of T. Let $g(t) \in F[t]$ be any polynomial such that $g(T) = 0$. Show that $f(t) \mid g(t)$. (Suggestion: By the division algorithm, we can write $g(t) = q(t)f(t) + r(t)$ with $\deg(r(t)) < \deg(f(t))$. It might help to take a look at Exercise 1 of the extra practice problems below.)

4. Let F be a field. We say a polynomial $f(t) \in F[t]$ is *irreducible* (over F) if it satisfies (both of) the following conditions: (i) the degree of $f(t)$ is ≥ 1, and (ii) whenever $f(t) = g(t)h(t)$ for

†We say a polynomial $f(t) \in F[t]$ is monic if its leading coefficient is 1.

polynomials $g(t), h(t) \in F[t]$, then $g(t)$ or $h(t)$ has degree zero (i.e. we cannot write $f(t)$ as a product of two elements of $F[t]$ with positive degree). For instance, $t^2 + 1$ is an irreducible polynomial in $\mathbb{R}[t]$. The only irreducible polynomials in $\mathbb{C}[t]$ are the degree 1 polynomials.

Let T be a linear operator on a finite-dimensional vector space over F.

(a) Show that for any $f(t) \in F[t]$, the kernel of $f(T)$ is T-invariant.

(b) Suppose $f(t) \in F[t]$ is an irreducible factor of the characteristic polynomial $p_T(t)$ of T (by being a factor we mean $f(t)$ divides $p_T(t)$). Show that either $\ker(f(T)) = 0$ or

$$\dim \ker(f(T)) \geq \deg(f(t)).$$

(Suggestion: Question 3 can be useful. Note: In fact, $f(T)$ cannot be injective. You can try to prove this (but it is not mandatory).)

5. Let V be an n-dimensional vector space over a field F. Suppose $T : V \to V$ is a linear map such that the characteristic polynomial $p_T(t)$ is irreducible.

(a) Show that the only T-invariant subspaces of V are 0 and V.

(b) Let v be a nonzero element of V. Show that $\{v, T(v), \ldots, T^{n-1}(v)\}$ is a basis of V.

(c) Suppose $A \in M_{4\times 4}(\mathbb{Q})$ has characteristic polynomial $p_A(t) = t^4 + t^3 + t^2 + t + 1$. Show that there exists a matrix $P \in M_{4\times 4}(\mathbb{Q})$ such that

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Note: For parts (b) and (c) you might want to wait until Tuesday. Otherwise, first read about T-cyclic subspaces on the bottom of page 313 and then read Theorem 5.22. For (c), you may take it for granted that $t^4 + t^3 + t^2 + t + 1$ is irreducible over $\mathbb{Q}$.)

Practice Problems: The following problems are for your practice. They are not to be handed in for grading.

From the textbook: exercises from 5.2 that use the word "multiplicity", exercises # 2, 4, 5, 7, 8, 19 of 5.4

Extra problems:

1. Let T be a linear operator on a finite-dimensional vector space over a field F. Let $f(t), g(t) \in F[t]$, $h(t) = f(t) + g(t)$ and $k(t) = f(t)g(t)$. Show that $h(T) = f(T) + g(T)$ and $k(T) = f(T) \circ g(T)$ (composition of $f(T)$ and $g(T)$). Conclude that the maps $f(T)$ and $g(T)$ commute.

2. Let V be a vector space over F and $T : V \to V$ a linear map. Let $f(t) \in F[t]$. Show that if $v \in V$ is an eigenvector of T with corresponding eigenvalue λ, then v is an eigenvector of $f(T)$ with corresponding eigenvalue $f(\lambda)$.

3. Suppose T is a diagonalizable operator on a finite-dimensional vector space V over a field F. Show that $f(T)$ is diagonalizable for every $f(t) \in F[t]$.

4. Prove Cayley-Hamilton for diagonalizable maps.

5. Let T be a linear operator on a vector space V over a field F. Let W be a T-invariant subspace of V. Show that W is $f(T)$-invariant for every $f(t) \in F[t]$.

6. (for those interested, will not be on the test/exam) Suppose F is a field of characteristic p. Show that if $p = 0$ or $p \nmid n$, then $t^n - 1$ has no repeated root in F (i.e. has no root of multiplicity > 1).

7. Let T be a linear operator on a finite-dimensional vector space V. Suppose there is a decomposition $V = \bigoplus_{i=1}^{k} W_i$, where the W_i are T-invariant. We denote the restriction of T to W_i by T_{W_i} (our usual notation).

(a) Let β_i be a basis of W_i. Let $\beta = \bigcup_i \beta_i$. Show that $[T]_\beta$ is the block diagonal matrix with the diagonal blocks $[T_{W_i}]_{\beta_i}$.

(b) Let $p_i(t)$ be the characteristic polynomial of T_{W_i}. Show that the characteristic polynomial of T equals $\prod_{i=1}^{k} p_i(t)$.

8. Let T be a linear operator on a finite-dimensional vector space V over a field F. For any $\lambda \in F$, define the *generalized eigenspace* of T corresponding to λ to be

$$K_\lambda := \{v \in V : (T - \lambda I)^m(v) = 0 \text{ for some positive integer } m\}.$$

Note that K_λ contains the eigenspace E_λ.

(a) Show that K_λ is T-invariant and that K_λ is nonzero if and only if λ is an eigenvalue of T.

(b) For an eigenvalue λ of T, let m_λ be the multiplicity of λ and $d_\lambda = \dim(K_\lambda)$. Show that the characteristic polynomial of the restriction of T to K_λ is $(-1)^{d_\lambda}(t-\lambda)^{d_\lambda}$. (Suggestion: Is the restriction of $T - \lambda I$ to K_λ nilpotent?)

(c) Let $\overline{T} : V/K_\lambda \to V/K_\lambda$ be the map induced by T on the quotient V/K_λ (see Problem 3 of Assignment 2). Show that λ is not an eigenvalue of $\overline{T}$. Conclude that $d_\lambda = m_\lambda$.

(d) Show that if $\lambda \neq \lambda'$, then $(T - \lambda' I)_{K_\lambda}$ (i.e. the restriction of $T - \lambda' I$ to K_λ) is injective. (Hint: What is the characteristic polynomial of $(T - \lambda' I)_{K_\lambda}$?)

(e) Show that the sum of the generalized eigenspaces of T corresponding to distinct eigenvalues is direct.

(f) Combine parts (c) and (e) to conclude that if the characteristic polynomial of T splits over F, then V decomposes as $\bigoplus_{\lambda} K_\lambda$, where the sum is over the eigenvalues of T.

9. (a) Let $A, P \in M_{n\times n}(\mathbb{C})$. Show that $e^{PAP^{-1}} = Pe^A P^{-1}$.

(b) Suppose $A \in M_{n\times n}(\mathbb{C})$ is diagonalizable over $\mathbb{C}$. Show that $e^{\mathrm{Tr}(A)} = \det(e^A)$ (where Tr stands for the trace). Note: The identity is also true for non-diagonalizable matrices.

10. (for interested students, will not be on the test/exam) Let F be a field. Let $U_n(F)$ be the set of upper triangular elements of $M_{n\times n}(F)$ with diagonal entries all equal to 1. Let $N_n(F)$ be the set of upper triangular nilpotent elements of $M_{n\times n}(F)$. Show that if F has characteristic zero, then the map $exp : N_n(F) \to U_n(F)$ sending $A \mapsto e^A$ (called the exponential map) is bijective. (Hint: Try to define the inverse of exp. It might be useful to try to borrow ideas from calculus.)