MAT247 Algebra II

Assignment 4

Solutions

1. Let T and S be linear operators on a vector space V such that $TS = ST$. Show that the kernel and image of S are T-invariant.

Solution: Let $v \in \ker(S)$. Then $S(T(v)) = T(S(v)) = T(0) = 0$, so that $T(v) \in \ker(S)$. Thus $\ker(S)$ is T-invariant.

Let $w \in \mathrm{Im}(S)$. Then $w = S(u)$ for some $u \in V$, and $T(w) = T(S(u)) = S(T(u)) \in \mathrm{Im}(S)$. Thus $\mathrm{Im}(S)$ is also T-invariant.

2. Let T be a linear operator on a nonzero finite-dimensional vector space V. Show that if V has no nontrivial T-invariant subspace (i.e. has no T-invariant subspaces other than 0 and V), then the characteristic polynomial of T is irreducible. Note: The converse statement is also true (and you proved it on the previous assignment).

Solution: Let $\dim(V) = n$. Suppose $p_T(t) = f(t)g(t)$. We need to show that either $f(t)$ or $g(t)$ has degree n. We have $p_T(T) = f(T) \circ g(T)$, so that by the Cayley-Hamilton theorem $f(T) \circ g(T) = 0$. It follows that either $f(T)$ or $g(T)$ is not injective (as otherwise, $f(T) \circ g(T)$ would be injective). Without loss of generality, say $f(T)$ is not injective. We claim that $f(t)$ has degree n. Indeed, let v be a nonzero element of $\ker(f(T))$. Let W be the T-cyclic subspace generated by v. Since V does not have any T-invariant subspaces other that zero and V, we must have $W = V$, so that (since $\dim(W) = n$) by Theorem 5.22(a), $\{v, T(v), \ldots, T^{n-1}(v)\}$ is linearly independent. Now note that if $\deg(f(t)) = m$, and $f(t) = \sum_{i=0}^{m} a_i t^i$ with $a_m \neq 0$ (note that $f(t)$ is not zero), then

$$0 = f(T)(v) = \sum_{i=0}^{m} a_i T^i(v).$$

If $m < n$, this contradicts the earlier conclusion that $\{v, T(v), \ldots, T^{n-1}(v)\}$ is linearly independent.

3. Let V be a vector space. Suppose V_i $(1 \leq i \leq k)$ are subspaces of V such that $V = \bigoplus_{i=1}^{k} V_i$. Let T be a linear operator on V such that each V_i is T-invariant. Show that

$$\ker(T) = \bigoplus_{i=1}^{k} (\ker(T) \cap V_i) = \bigoplus_{i=1}^{k} \ker(T_{V_i})$$

and

$$\mathrm{Im}(T) = \bigoplus_{i=1}^{k} (\mathrm{Im}(T) \cap V_i) = \bigoplus_{i=1}^{k} \mathrm{Im}(T_{V_i}).$$

(As usual, $T_W : W \to W$ denotes the restriction of T to a T-invariant subspace W of V.)

Solution: Let us first focus on the assertions regarding kernels. Since the sum of the V_i is direct, it is clear that the sum of the subspaces $\ker(T) \cap V_i$ is direct. Thus to prove the first equality it is enough to show that

$$\ker(T) = \sum_{i=1}^{k} (\ker(T) \cap V_i). \tag{1}$$

Let $v \in \ker(T)$. Since $V = \bigoplus_i V_i$, we can express v uniquely as $v = \sum_i v_i$ for vectors $v_i \in V_i$. (That we can express v as a sum of vectors in the V_i is because $V = \sum_i V_i$, and the uniqueness follows easily from property (ii) of a direct sum from Problem 2 of Assignment 2.) Then $T(v) = \sum_i T(v_i)$. Since $v \in \ker(T)$, we see $\sum_i T(v_i) = 0$. Since each V_i is T-invariant, we have $T(v_i) \in V_i$. Since the sum of the V_i is direct, it follows that $T(v_i) = 0$ for all i, so that $v_i \in \ker(T) \cap V_i$. This completes the proof of (1).

The equality

$$\bigoplus_{i=1}^{k} (\ker(T) \cap V_i) = \bigoplus_{i=1}^{k} \ker(T_{V_i})$$

is clear because $\ker(T) \cap V_i = \ker(T_{V_i})$.

We now prove the assertions for images. To prove

$$\mathrm{Im}(T) = \bigoplus_{i=1}^{k} (\mathrm{Im}(T) \cap V_i),$$

it is enough to show that

$$\mathrm{Im}(T) = \sum_{i=1}^{k} (\mathrm{Im}(T) \cap V_i) \tag{2}$$

(as the sum of $\mathrm{Im}(T) \cap V_i$ is certainly direct). Let $v \in \mathrm{Im}(T)$. Then $v = T(u)$ for some $u \in V$. Since $V = \bigoplus_i V_i$, we can write $u = \sum_i u_i$ for unique vectors $u_i \in V_i$. Then $v = T(u) = \sum_i T(u_i)$. Since each V_i is T-invariant, $T(u_i) \in V_i$, and hence in $V_i \cap \mathrm{Im}(T)$. This proves (2).

It remains to show that

$$\bigoplus_{i=1}^{k} (\mathrm{Im}(T) \cap V_i) = \bigoplus_{i=1}^{k} \mathrm{Im}(T_{V_i}).$$

It is enough to show that

$$\mathrm{Im}(T) \cap V_i = \mathrm{Im}(T_{V_i})$$

for any i. The inclusion $\mathrm{Im}(T) \cap V_i \supset \mathrm{Im}(T_{V_i})$ is clear. We shall show $\mathrm{Im}(T) \cap V_i \subset \mathrm{Im}(T_{V_i})$. Let $v \in \mathrm{Im}(T) \cap V_i$. Then $v = T(u)$ for some $u \in V$. Write $u = \sum_j u_j$ with $u_j \in V_j$. Then $v = \sum_j T(u_j)$. Since each V_j is T-invariant, we have $T(u_j) \in V_j$. Since $v \in V_i$ and the sum of the V_j is direct, it follows that $v = T(u_i)$ and $T(u_j) = 0$ for $j \neq i$. In particular, $v \in T(V_i) = \mathrm{Im}(T_{V_i})$, as desired.

4. Let T be a nilponent linear operator on a (possibly infinite-dimensional) vector space V. Suppose the nilpotency index of T is k. (That is, k is the smallest non-negative integer such that $T^k = 0$.) Show that if $0 \leq i < k$, then $\mathrm{Im}(T^{i+1}) \subsetneq \mathrm{Im}(T^i)$ and $\ker(T^i) \subsetneq \ker(T^{i+1})$. Suggestion: It is useful to note that $T(\mathrm{Im}(T^i)) = \mathrm{Im}(T^{i+1})$ and $\mathrm{Im}(T^i) \subset \ker(T^{k-i})$.

Solution: It is clear that $\mathrm{Im}(T^{i+1}) \subset \mathrm{Im}(T^i)$ for all i (as $T^{i+1}(v) = T^i(T(v))$) for any v). We need to show that for $0 \leq i < k$, $\mathrm{Im}(T^{i+1}) \neq \mathrm{Im}(T^i)$. Suppose for some $0 \leq i < k$ we have

$\mathrm{Im}(T^{i+1}) = \mathrm{Im}(T^i)$. Then since $\mathrm{Im}(f \circ g) = f(\mathrm{Im}(g))$ for any composable functions f and g, applying powers of T we see that $\mathrm{Im}(T^{j+1}) = \mathrm{Im}(T^j)$ for all $j \geq i$, and hence for all such j, $\mathrm{Im}(T^j) = \mathrm{Im}(T^i)$. In particular, $\mathrm{Im}(T^i) = \mathrm{Im}(T^k) = 0$, which contradicts the defining property of k.

As for the assertion regarding kernels, it is clear that for every i, $\ker(T^i) \subset \ker(T^{i+1})$. Let $0 \leq i < k$. We shall show that $\ker(T^i) \neq \ker(T^{i+1})$. Indeed, suppose $\ker(T^i) = \ker(T^{i+1})$. Since $T^k = 0$ and $i < k$, we have $\mathrm{Im}(T^{k-i-1)}) \subset \ker(T^{i+1})$, so that $\mathrm{Im}(T^{k-i-1}) \subset \ker(T^i)$. But then $T^{k-1} = T^i T^{k-i-1} = 0$, again contradicting the defining property of k.

5. Let

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

(a) Find the dimensions of the nullspaces of $(A - 2I)^k$ for $k = 1, 2, \ldots$.
(b) Show that A is not similar to the matrix

$$B = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Hint: If A and B are similar, then $f(A)$ and $f(B)$ are similar for any polynomial $f(t)$. Similar matrices have the same nullity (for if $C = P^{-1}DP$, then we have an isomorphism $N(C) \to N(D)$ given by $x \mapsto Px$).

Solution: (a) $\dim(N(A - 2I)) = 4$, $\dim(N(A - 2I)^2) = 7$, $\dim(N(A - 2I)^3) = 8$, $\dim(N(A - 2I)^k) = 9$ for $k \geq 4$

(b) We have $\dim(N(B - 2I)^3) = 9 \neq \dim(N(A - 2I)^3)$, so A and B are not similar. (See the explanation in the hint.)

6. (reading assignment) Read Theorems E2, E8 and E9 and their proofs (and the relevant definitions) from Appendix E.