MAT247 Algebra II

Assignment 4

Due Saturday Feb 9 at 11:59 pm
(to be submitted on Crowdmark)

Please write your solutions neatly and clearly. Note that due to time limitations, some questions may not be graded.

1. Let T and S be linear operators on a vector space V such that $TS = ST$. Show that the kernel and image of S are T-invariant.

2. Let T be a linear operator on a nonzero finite-dimensional vector space V. Show that if V has no nontrivial T-invariant subspace (i.e. has no T-invariant subspaces other than 0 and V), then the characteristic polynomial of T is irreducible. Note: The converse statement is also true (and you proved it on the previous assignment).

3. Let V be a vector space. Suppose V_i ($1 \le i \le k$) are subspaces of V such that $V = \bigoplus_{i=1}^{k} V_i$. Let T be a linear operator on V such that each V_i is T-invariant. Show that

$$\ker(T) = \bigoplus_{i=1}^{k} (\ker(T) \cap V_i) \; = \bigoplus_{i=1}^{k} \ker(T_{V_i})$$

and

$$\mathrm{Im}(T) = \bigoplus_{i=1}^{k} (\mathrm{Im}(T) \cap V_i) \; = \bigoplus_{i=1}^{k} \mathrm{Im}(T_{V_i}).$$

(As usual, $T_W : W \to W$ denotes the restriction of T to a T-invariant subspace W of V.)

4. Let T be a nilponent linear operator on a (possibly infinite-dimensional) vector space V. Suppose the nilpotency index of T is k. (That is, k is the smallest nonnegative integer such that $T^k = 0$.) Show that if $0 \le i < k$, then $\mathrm{Im}(T^{i+1}) \subsetneq \mathrm{Im}(T^i)$ and $\ker(T^i) \subsetneq \ker(T^{i+1})$. Suggestion: It is useful to note that $T(\mathrm{Im}(T^i)) = \mathrm{Im}(T^{i+1})$ and $\mathrm{Im}(T^i) \subset \ker(T^{k-i})$.

5. Let

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

(a) Find the dimensions of the nullspaces of $(A - 2I)^k$ for $k = 1, 2, \ldots$.

(b) Show that A is not similar to the matrix

$$B = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Hint: If A and B are similar, then $f(A)$ and $f(B)$ are similar for any polynomial $f(t)$. Similar matrices have the same nullity (for if $C = P^{-1}DP$, then we have an isomorphism $N(C) \to N(D)$ given by $x \mapsto Px$).

6. (reading assignment) Read Theorems E2, E8 and E9 and their proofs (and the relevant definitions) from Appendix E.

Practice Problems: The following problems are for your practice. They are not to be handed in for grading.

From the textbook: remaining exercises of 5.4, in particular # 1, 5, 11, 13, 14, 15, 16 (you may need to use the unique factorization theorem from Appendix E), 17, 18, 20-25, 29, 30, 33, 36-38, 39, 41, 42

Extra problems:

1. Let T be a linear operator on a finite-dimensional vector space V over a field F. Let $f(t) \in F[t]$ be a polynomial of degree ≥ 1 which divides the characteristic polynomial of T. Show that $\det(f(T)) = 0$ (equivalently, that $\ker(f(T)) \neq 0$). (Hint: Here is one approach: First, using the fact that any polynomial of positive degree is a product of irreducible polynomials, argue that without loss of generality we may assume $f(t)$ is irreducible. Now argue by induction on the dimension of V. Consider two cases, based on whether V has any nontrivial T-invariant subspaces. Theorem E8 and quotient vector spaces can be helpful.)

2. Let F be a field and $f(t), g(t) \in F[t]$ relatively prime (or coprime). Let T be a linear operator on a vector space V over F. Show that

$$\ker(f(T)g(T)) = \ker(f(T)) \oplus \ker(g(T)).$$

(Hint: Theorem E2.)

3. (a) Let F be a field. Suppose $f(t), g_1(t), \ldots, g_k(t) \in F[t]$ are such that $f(t)$ is relatively prime to each $g_i(t)$. Show that $f(t)$ is relatively prime to $\prod_{i=1}^{k} g_i(t)$. (Argue by induction on k.)

(b) Let $f_1(t), \ldots, f_k(t) \in F[t]$ be pairwise coprime. Let T be a linear operator on a vector space V over F. Show that

$$\ker(f_1(T) \cdots f_k(T)) = \bigoplus_{i=1}^{k} \ker(f_i(T)).$$

(Is it true that if $V = U_1 \oplus W$ and $W = U_2 \oplus \cdots \oplus U_k$, then $V = U_1 \oplus \cdots \oplus U_k$?)

4. Let V be a vector space. Suppose V_i ($1 \leq i \leq k$) is a subspace of V and $V = \bigoplus_{i=1}^{k} V_i$. Then for each $v \in V$, there exist unique vectors $v_i \in V_i$ ($1 \leq i \leq k$) such that $v = \sum_{i=1}^{k} v_i$. (Existence follows from $V = \sum_{i=1}^{k} V_i$ and uniqueness follows from the fact that the sum is direct, by the property of direct sum given in Problem 2(ii) of Assignment 2.) We refer to v_i as the component of v in V_i (or the V_i-component of v).

For each $1 \leq i \leq k$, let $\pi_i : V \to V$ (called the projection map to V_i relative to the decomposition $V = \bigoplus_{j=1}^{k} V_j$) be the map that sends $v \in V$ to its component in V_i.

(a) Show that the maps π_i satisfy the following properties: (i) they are linear, (ii) $\ker(\pi_i) = \sum_{j \neq i} V_j$ and $\mathrm{Im}(\pi_i) = V_i$, (iii) $\pi_{i_{V_i}} = I_{V_i}$ (identity map on V_i), (iv) $\pi_i^2 = \pi_i$, (v) $\sum_{i=1}^{k} \pi_i = I_V$, and (vi) $\pi_i \pi_j = 0$ if $j \neq i$.

(b) Let $p_i : V \to V$ $(1 \leq i \leq k)$ be linear maps such that $\sum_{i=1}^{k} p_i = I_V$ and $p_i p_j = 0$ if $j \neq i$. Show that $V = \bigoplus_{i=1}^{k} \mathrm{Im}(p_i)$ and that p_i is the projection map to $\mathrm{Im}(V_i)$ relative to this decomposition.

5. Let V be a vector space, W a subspace of V and T a linear operator on V. Show that the following statements are equivalent:

(i) W is T-invariant.

(ii) If U is any complementary subspace for W and π_W is the projection map to W relative to the decomposition $V = W \oplus U$, then the two maps $\pi_W T$ and $T\pi_W$ agree on W. (Recall that a complementary subspace for W means a subspace U of V such that $V = W \oplus U$.)

(iii) There exists a complementary subspace U for W such that $\pi_W T = T\pi_W$ on W.