MAT247 Algebra II

Assignment 5

Solutions

1. Find the Jordan canonical form and a Jordan basis for the map or matrix given in each part below.

(a) Let V be the real vector space spanned by the polynomials $x^i y^j$ (in two variables) with $i + j \leq 3$. Let $T : V \to V$ be the map $D_x + D_y$, where D_x and D_y respectively denote differentiation with respect to x and y. (Thus, $D_x(xy) = y$ and $D_y(xy^2) = 2xy$.)

(b) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ over $\mathbb{Q}$

Solution: (a) The operator T is nilpotent so its characterictic polynomial splits and its only eigenvalue is zero and $K_0 = V$. We have

$$\begin{aligned} \mathrm{Im}(T) &= span\{1, x, y, x^2, xy, y^2\}, \\ \mathrm{Im}(T^2) &= span\{1, x, y\} \\ \mathrm{Im}(T^3) &= span\{1\} \\ T^4 &= 0. \end{aligned}$$

Thus the longest cycle for eigenvalue zero has length 4. Moreover, since the number of cycles of length at least r is given by $\dim(\mathrm{Im}(T^{r-1})) - \dim(\mathrm{Im}(T^r))$, the number of cycles of lengths at least 4, 3, 2, 1 is respectively 1, 2, 3, and 4. Thus the number of cycles of lengths 4,3,2,1 is respectively 1,1,1,1. Denoting the $r \times r$ Jordan block with λ on the diagonal by $J_{\lambda,r}$, the Jordan canonical form is thus

$$J = \begin{pmatrix} J_{0,4} & & & \\ & J_{0,3} & & \\ & & J_{0,2} & \\ & & & J_{0,1} \end{pmatrix}.$$

We now proceed to find a corresponding Jordan basis, i.e. a basis β of V such that $[T]_\beta = J$. This will be a union $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are disjoint cycles of generalized eigenvectors of lengths respectively 4,3,2,1. We shall follow the general procedure we gave for calculating a basis of a generalized eigenspace that is a union of disjoint cycles. Note that

$$E_0 = \ker(T) = span\{1, x - y, (x - y)^2, (x - y)^3\}.$$

Since $\mathrm{Im}(T^3) \cap E_0 = span\{1\}$, we take 1 as the initial vector of γ_1. In search for a possible end vector (or generator) for γ_1, we solve $T^3(v) = 1$. We see $v = \frac{1}{6}x^3$ is a solution. We now take

$$\gamma_1 = \{T^3(v), T^2(v), T(v), v\} = \{1,\ x,\ \frac{1}{2}x^2,\ \frac{1}{6}x^3\}.$$

Next, we find γ_2. We have $\mathrm{Im}(T^2) \cap E_0 = span\{1, x - y\}$ (note that before finding this we already know by part (c) of the proposition we proved in class that $\mathrm{Im}(T^2) \cap E_0$ is 2-dimensional). We take $x - y$ as the initial vector of γ_3. A solution to $T^2(v) = x - y$ is $v = \frac{1}{6}(x^3 - y^3)$; we take this as the end vector for γ_2, so that

$$\gamma_2 = \{x - y,\ \frac{1}{2}(x^2 - y^2),\ \frac{1}{6}(x^3 - y^3)\}.$$

We have $\mathrm{Im}(T) \cap E_0 = \mathrm{span}\{1, x-y, (x-y)^2\}$, so that we can take $(x-y)^2$ as the initial vector for γ_3. Solving $T(v) = (x-y)^3$, we see a solution is $\frac{1}{2}(x^3 - x^2y - xy^2 + y^3)$, so that we can take

$$\gamma_3 = \{(x-y)^3,\ \frac{1}{2}(x^3 - x^2y - xy^2 + y^3)\}.$$

Finally, since $\{1, x-y, (x-y)^2, (x-y)^3\}$ is a basis for E_0, we can take $\gamma_4 = \{(x-y)^3\}$.

(b) The characteristic polynomial is $-(t-1)^3(t-2)$, which splits over $\mathbb{Q}$. Hence the matrix has a Jordan canonical form over $\mathbb{Q}$. The eigenvalues are 1 and 2. Since the dimension of a generalized eigenspace is equal to the multiplicity of the corresponding eigenvalue, we know $\dim(K_1) = 3$ and $\dim(K_2) = 1$. It follows from the latter that the Jordan form has only one block corresponding to eigenvalue 2, which is 1×1. As for K_1, there are three possibilities for the dot diagram:

$$\begin{array}{c}\bullet\\ \bullet\\ \bullet\end{array}, \qquad \begin{array}{cc}\bullet & \bullet\\ \bullet & \end{array}, \qquad \text{or} \qquad \begin{array}{ccc}\bullet & \bullet & \bullet\end{array}.$$

These are distinguished by the number of cycles (= number of columns) in them. We easily see $E_1 = \mathrm{span}\{(1\ 0\ 0\ 0)^t\}$, so that there will be only one cycle. Thus the dot diagram is the first one above. With the same notation as in the solution to part (a), the Jordan canonical form of A is thus

$$J = \begin{pmatrix} J_{1,3} & 0 \\ 0 & J_{2,1} \end{pmatrix}.$$

We now proceed to find a Jordan basis of $\mathbb{Q}^4$ for A. A basis for $K_2 = E_2$ is $\{(1\ 1\ -1\ 2)^t\}$. Turning our attention to K_1, we already know that our basis for K_1 will consist of a single cycle of length 3. The initial vector is an element of $E_1 = \mathrm{span}\{(1\ 0\ 0\ 0)^t\}$. We take the initial vector to be $e_1 = (1\ 0\ 0\ 0)^t$. In search for a possible end vector for the cycle, we solve $(A-I)^2x = e_1$; a solution is $x = \frac{1}{2}(0\ 0\ 1\ 0)^t$. We take γ to be the cycle of generalized eigenvectors for eigenvalue 1 generalized by this x:

$$\gamma = \{(A-I)^2x, (A-I)x, x\} = \{(1\ 0\ 0\ 0)^t,\ \frac{1}{2}(1\ 2\ 0\ 0)^t,\ \frac{1}{2}(0\ 0\ 1\ 0)^t\}.$$

Thus

$$\beta = \gamma \cup \{(1\ 1\ -1\ 2)^t\}$$

is a Jordan basis for A. We have $[L_A]_\beta = J$.

2. Let K be a field and $A, B \in M_{n\times n}(K)$. Suppose A and B are similar over K; in other words, suppose there exists a matrix $P \in M_{n\times n}(K)$ such that $A = PBP^{-1}$. Let F be a subfield of K which contains the entries of A and B. Assuming that the characteristic polynomial of A (which is the same as that of B) splits over F, show that A and B are also similar over F. (Hint: Uniqueness of Jordan canonical form. Remark: We will see later that the statement it true regardless of whether or not the characteristic polynomial splits over F or even K.)

Solution: Let us first make a few remarks:

(1) One can see using the change of basis formula that given any $X, Y \in M_{n\times n}(F)$, the matrices X and Y are similar over F if and only if there exists a basis β of F^n such that $Y = [L_X]_\beta$. It follows that a Jordan matrix $J \in M_{n\times n}(F)$ is a Jordan canonical form for X over F (that is, by definition, for the map $L_X : F^n \to F^n$) if and only if J is similar to X over F.

(2) Let J and J' be Jordan matrices in $M_{n\times n}(F)$. Then they are similar over F if and only if one of them is obtained from the other by a permutation of the Jordan blocks. That is, if say

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{pmatrix}, \; J' = \begin{pmatrix} J'_1 & & \\ & \ddots & \\ & & J_{r'} \end{pmatrix} \in M_{n\times n}(F), \tag{1}$$

with the J_i and J'_j Jordan blocks, then J and J' are similar over F if and only if $r = r'$ and there exists a permutation $\sigma \in S_r$ such that $J'_i = J_{\sigma(i)}$ for all $1 \le i \le r$. Indeed, if J and J' are similar over F, by the previous remark, J' is a Jordan canonical form for J. Since clearly J is also a Jordan canonical form for J, it follows from the uniqueness of Jordan canonical form that J' is obtained from J by permuting the Jordan blocks. Conversely, let ε be the standard ordered basis of F^n. For J is as in Eq. (1), we break up the ordered basis ε as a disjoint union $\gamma_1 \cup \ldots \cup \gamma_r$ with $|\gamma_i|$ the order of J_i. Then given a permutation $\sigma \in S_r$, if we take β to be the (ordered) basis $\gamma_{\sigma(1)} \cup \ldots \cup \gamma_{\sigma(r)}$ of F^n, then $[L_J]_\beta$ is the Jordan matrix with Jordan blocks $J_{\sigma(1)}, \ldots, J_{\sigma(r)}$, in that order.

(3) Let J and J' be Jordan matrices in $M_{n\times n}(F)$ (and hence in $M_{n\times n}(K)$). It follows from the previous remark that if J and J' are similar over K, then then they are also similar over F. Indeed, similarity of J and J' over K or over F are both equivalent to the matrices J and J' being obtained from one another by permuting the Jordan blocks (the latter condition is not sensitive to the field).

We know return to the problem. For $X, Y \in M_{n\times n}(F)$, we shall write $X \sim_F Y$ to indicate that the matrices X, Y are similar over F. It is clear that $X \sim_F Y$ implies $X \sim_K Y$. Since the characteristic polynomial of A splits over F, A has a Jordan canonical form J_A over F. By Remark (1) above, $A \sim_F J_A$. Similarly, B has a Jordan canonical form J_B over F and we have $B \sim_F J_B$. We then clearly have $A \sim_K J_A$ and $B \sim_K J_B$. Since $A \sim_K B$ and $\sim_K$ is an equivalence relation on $M_{n\times n}(K)$, it follows $J_A \sim_K J_B$. By Remark (3) above, $J_A \sim_F J_B$. Since $\sim_F$ is an equivalence relation, $A \sim_F B$.

3. Determine which of the following eight matrices are similar to each other.

$$A = \begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \; B = \begin{pmatrix} 0&0&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \; C = \begin{pmatrix} 0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \; D = \begin{pmatrix} 0&1&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}$$

$$E = \begin{pmatrix} 0&1&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \; F = \begin{pmatrix} 0&1&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&0 \end{pmatrix}, \; G = \begin{pmatrix} 0&1&0&5\\0&0&0&0\\0&1&0&1\\0&0&0&0 \end{pmatrix}, \; H = \begin{pmatrix} 0&1&0&0\\1&0&1&0\\0&0&0&0\\0&1&0&0 \end{pmatrix}$$

Solution: We easily see that the the characteristic polynomials of the matrices $A, \ldots, G$ are all t^4, whereas the characteristic polynomial of H is $t^2(t^2-1)$. Thus H is not similar to any of the other given matrices (because similar matrices have the same characteristic polynomial). As for $A, \ldots, G$, we find their Jordan canonical forms. Denoting the $r \times r$ Jordan block corresponding to zero by $J_{0,r}$, the matrices A, B and D have a Jordan canonical form

$$J = \begin{pmatrix} J_{0,2} & & \\ & 0 & \\ & & 0 \end{pmatrix}$$

(as each has nullity (= dimension of E_0 = number of cycles in a Jordan basis of $K_0 = \mathbb{F}^4$, where $\mathbb{F}$ is the field) three and that completely determines the dot diagram here). The matrices C and G have a Jordan canonical form

$$J' = \begin{pmatrix} J_{0,2} & \\ & J_{0,2} \end{pmatrix}$$

(as each has nullity 2 and $C^2 = G^2 = 0$, so the longest cycle has length 2). Finally, E and F have a Jordan canonical form

$$J'' = \begin{pmatrix} J_{0,3} & \\ & 0 \end{pmatrix}$$

(as they have nullity 2 and their square is not zero, so that they must have a cycle of length 3). It follows that A, B, D are similar to J and hence one another, while C and G are similar to J' and hence each other, and finally E and F are similar to J'', and hence one another. Moreover, no matrix from one of these families is similar to a matrix from another, as no two of J, J' and J'' are similar (see Remark (2) in the solution to the previous problem).

4. Find a 5-th root for the matrix

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$$

in $M_{2\times 2}(\mathbb{C})$, if there is one. In other words, find a matrix $X \in M_{2\times 2}(\mathbb{C})$ such that $X^5 = A$, if such X exists. Hint: Find the Jordan canonical form of A and a Jordan basis for it. Also note that

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}.$$

Solution: Since $\mathbb{C}$ is algebraically closed, A has a Jordan canonical form J over $\mathbb{C}$. We first find J and a matrix P such that $P^{-1}AP = J$. If we can find a 5-th root Y for J, then

$$(PYP^{-1})^5 = PY^5P^{-1} = PJP^{-1} = A,$$

so that $X = PYP^{-1}$ will be a 5-th root for A.

The characteristic polynomial of A is $(t-2)^2$ and $\dim E_2 = 1$. It follows that a Jordan basis for A will consist of only one cycle of generalized eigenvalues for eigenvalue 2, so that the Jordan canonical form is

$$J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}.$$

By the observation given in the hint,

$$Y = \sqrt[5]{2}\begin{pmatrix} 1 & \frac{1}{10} \\ 0 & 1 \end{pmatrix}$$

is a 5-th root of J. It remains to find a P such that $P^{-1}AP = J$. This amounts to finding a Jordan basis for A. A basis for E_2 is $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$. We thus take $b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as the initial vector of the cycle, and solve $(A - 2I)x = b$. A solution is $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then

$$\beta = \{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$$

is a Jordan basis for A, and $[L_A]_\beta = J$. By the change of basis formula, setting

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$

we have $P^{-1}AP = [L_A]_\beta = J$. By the remark made in the beginning of the solution, $X = PYP^{-1}$ (with P and Y as above) is a 5-th root for A.

5. Find an explicit (i.e. non-recursive) formula for the n-th term of the sequence defined by $a_1 = 1$, $a_2 = 5$, and $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 3$.

Solution: We first formulate the problem in terms of linear algebra. Set $x_n = (a_n\ a_{n+1})^t$. We have $x_1 = (1\ 5)^t$, and by the recursive formula for the a_n, for $n \geq 2$,

$$x_n = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix} x_{n-1}.$$

Call the 2×2 matrix above A. Then we have $x_n = A^{n-1}x_1$. Our goal will be to calculate A^{n-1} using techniques from linear algebra. Then we'll have x_n, and the first entry of x_n is what we are looking for.

To calculate the powers of A, we find a Jordan canonical form and corresponding Jordan basis for A. (If we have to, we work over $\mathbb{C}$, where A is guaranteed to have a Jordan canonical form.) The characteristic polynomial of A is $t^2 - 4t + 4 = (t-2)^2$. Leaving the derivations to the reader, we find that the Jordan canonical form is

$$J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix},$$

and that with

$$P = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

we have $J = P^{-1}AP$. We then have

$$A^{n-1} = PJ^{n-1}P^{-1} = 2^{n-1} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{n-1}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = 2^{n-1} \begin{pmatrix} 2-n & \frac{n-1}{2} \\ 2-2n & n \end{pmatrix}.$$

Thus

$$x_n = A^{n-1} \begin{pmatrix} 1 \\ 5 \end{pmatrix} = 2^{n-1} \begin{pmatrix} \frac{3n-1}{2} \\ * \end{pmatrix},$$

so that

$$a_n = 2^{n-2}(3n-1).$$