MAT247 Algebra II

Assignment 5

Due Friday Feb 22 at 11:59 pm
(to be submitted on Crowdmark)

Please write your solutions neatly and clearly. Note that due to time limitations, some questions may not be graded.

1. Find the Jordan canonical form and a Jordan basis for the map or matrix given in each part below.

(a) Let V be the real vector space spanned by the polynomials $x^i y^j$ (in two variables) with $i + j \leq 3$. Let $T : V \to V$ be the map $D_x + D_y$, where D_x and D_y respectively denote differentiation with respect to x and y. (Thus, $D_x(xy) = y$ and $D_y(xy^2) = 2xy$.)

(b) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ over $\mathbb{Q}$

2. Let K be a field and $A, B \in M_{n\times n}(K)$. Suppose A and B are similar over K; in other words, suppose there exists a matrix $P \in M_{n\times n}(K)$ such that $A = PBP^{-1}$. Let F be a subfield of K which contains the entries of A and B. Assuming that the characteristic polynomial of A (which is the same as that of B) splits over F, show that A and B are also similar over F. (Hint: Uniqueness of Jordan canonical form. Remark: We will see later that the statement it true regardless of whether or not the characteristic polynomial splits over F or even K.)

3. Determine which of the following eight matrices are similar to each other.

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; D = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; F = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; G = \begin{pmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

4. Find a 5-th root for the matrix

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$$

in $M_{2\times 2}(\mathbb{C})$, if there is one. In other words, find a matrix $X \in M_{2\times 2}(\mathbb{C})$ such that $X^5 = A$, if such X exists. Hint: Find the Jordan canonical form of A and a Jordan basis for it. Also note that

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}.$$

5. Find an explicit (i.e. non-recursive) formula for the n-th term of the sequence defined by $a_1 = 1$, $a_2 = 5$, and $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 3$.

Practice Problems: The following problems are for your practice. They are not to be handed in for grading.

From the textbook: exercises # 1,2,3, 4, 7e of 7.1, exercises # 1, 2, 3, 4, 5, 6 of 7.2

Extra problems:

1. Let λ be an eigenvalue of a linear operator T on a finite-dimensional vector space V. Show that

$$\sum_{r=0}^{\infty} \dim\left(E_\lambda \cap \mathrm{Im}(T-\lambda I)^r\right) = \dim K_\lambda.$$

(Hint: We know there exists a basis of K_λ which is a union of disjoint cycles of generalized eigenvectors corresponding to λ. We also know that the number of cycles of length at least r among them equals $\dim\left(E_\lambda \cap \mathrm{Im}(T-\lambda I)^{r-1}\right)$.)

2. Find the Jordan canonical form and a corresponding Jordan basis for the map/matrix given in each part below, if a Jordan canonical form exists.

(a) $T : V \to V$, where V is the complex vector space spanned by the polynomials $x^i y^j$ with $i+j \leq 2$, and T is the map xD_x, where D_x is differentiation with respect to x.

(b) $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ over a field F of characteristic 3

3. Let T be a linear operator on a vector space over a field of characteristic $\neq 2$. Suppose the characteristic polynomial of T is $t^5(t+1)^4(t+2)$. Answer the following questions.

(a) Find the dimension of each generalized eigenspace of T. What can you say about the dimension of each eigenspace?

(b) What is $\dim\ker(T+I)^4$? Do we have enough information to determine $\dim\ker(T+I)^3$?

(c) Suppose that in the Jordan canonical form of T there are 3 Jordan blocks corresponding to eigenvalue -1. What is $\dim E_{-1}$? What is $\dim(\ker(T+I)^2)$? (List all possible values, if there is more than one.)

(d) Suppose that in the Jordan canonical form of T there are 3 Jordan blocks corresponding to eigenvalue 0. What is $\dim E_0$? What is $\dim(\ker(T^2))$? (List all possible values, if there is more than one.)

4. (a) Let $J_{\lambda,k}$ be the $k \times k$ Jordan block corresponding to λ. Find $J_{\lambda,k}^n$ for $n \geq 1$. (Hint: Write $J_{\lambda,k} = \lambda I + J_{0,k}$. Powers of $J_{0,k}$ can be easily computed.)

(b) Describe how you would go about calculating high powers of a matrix with entries in $\mathbb{C}$ (or more generally, any field, assuming the characteristic polynomial of the given matrix splits over the field).

5. (for interested students, won't be on the test or exam) Show that any square matrix with complex entries with $E_0 = K_0$ has an n-th root for every positive integer n.

6. Let V be a finite-dimensional vector space. Show that any linear operator $T : V \to V$ and its dual $T^\vee : V^\vee \to V^\vee$ have the same Jordan canonical form. (Hint: One approach is to construct a Jordan basis of $T^\vee$ from a Jordan basis of T.)

7. Determine if each statement below is true or false. Throughout, T is a linear operator on a finite-dimensional vector space V over a field F, and for any $\lambda \in F$, E_λ (resp. K_λ) denotes the eigenspace (resp. generalized eigenspace) of T corresponding to λ.

(a) Every T-invariant subspace of V has a T-invariant complementary subspace.

(b) T is diagonalizable if and only if for every $\lambda \in F$, $E_\lambda = K_\lambda$.

(c) If the characteristic polynomial of T splits over F, then T is diagonalizable if and only if for every $\lambda \in F$, $E_\lambda = K_\lambda$.
(d) If $F = \mathbb{C}$, then T has a Jordan canonical form.
(e) If λ is an eigenvalue of T of multiplicity m, then there exists a basis of V with respect to which the matrix of T has the form

$$\begin{pmatrix} J & 0 \\ 0 & A \end{pmatrix},$$

where J is an $m \times m$ Jordan matrix with diagonal entries λ, and A a square matrix. (Hint: Problem 2 of the extra practice problems in Assignment 4)
(f) If λ is an eigenvalue of T of multiplicity m, then for any $k \leq m$, there exists a basis of V with respect to which the matrix of T has the form

$$\begin{pmatrix} J & 0 \\ 0 & A \end{pmatrix},$$

where J is a $k \times k$ Jordan matrix with diagonal entries λ, and A a square matrix.
(g) If λ is an eigenvalue of T of multiplicity m, then for any $k \leq m$, there exists a basis of V with respect to which the matrix of T has the form

$$\begin{pmatrix} J & * \\ 0 & A \end{pmatrix},$$

where J is a $k \times k$ Jordan matrix with diagonal entries λ, and A a square matrix.

8. Let T be an operator on an n-dimensional vector space V over a field F. Suppose there exists a field K containing F such that the characteristic polynomial $p_T(t)$ of T splits over K, say

$$p_T(t) = (-1)^n \prod_{i=1}^{n} (t - \lambda_i),$$

where $\lambda_1, \ldots, \lambda_n \in K$ (not necessarily distinct)†. Show that for every positive integer r, the characteristic polynomial of T^r is

$$(-1)^n \prod_{i=1}^{n} (t - \lambda_i^r).$$

(Hint: To get around the fact that the characteristic polynomial may not split over F, extend the scalars as in the last problem below.)

On earlier material:

9. true or false: If $V = U \oplus W$, then for every subspace V' of V, we have $V' = (V' \cap U) \oplus (V' \cap W)$.
10. Let $T : V \to W$ be a linear map. Let $T^\vee : W^\vee \to V^\vee$ be the dual map. Show that T is surjective (resp. injective) if and only if $T^\vee$ is injective (resp. surjective).
11. Let V be a vector space. For any subspace U of V, define

$$U^\perp := \{f \in V^\vee : f \text{ vanishes on } U\}.$$

(a) Show that for any subspace U of V, $U^\perp$ is a subspace of $V^\vee$.
(b) Show that if $V = U \oplus W$, then $V^\vee = U^\perp \oplus W^\perp$.
(c) Given a subspace U of V, construct an isomorphism $(V/U)^\vee \to U^\perp$. (Hint: Let $\eta : V \to V/U$ be the quotient map (sending $v \mapsto v + U$). What is the image of $\eta^\vee : (V/U)^\vee \to V^\vee$?)

†You will see later in your abstract algebra course that such K always exists.

(d) Show that if V is finite-dimensional, then $\dim(U^{\perp}) = \dim(V) - \dim(U)$ for every subspace U of V.

(e) Show that if V is finite-dimensional, any subspace $\mathcal{F}$ of $V^{\vee}$ equals $U^{\perp}$ for some subspace U of V. (Hint: There is a good candidate for U.)

(f) Show that if V is not finite-dimensional, then there exists a subspace $\mathcal{F}$ of $V^{\vee}$ which is not of the form $U^{\perp}$ for any subspace U of V.

12. Let T be an invertible operator on a finite-dimensional vector space V over F. Show that there exists a polynomial $f(t) \in F[t]$ such that $T^{-1} = f(T)$. (Hint: Cayley-Hamilton.)

13. Suppose T is an operator on an n-dimensional vector space V over a field F. Let $f(t)$ (resp. $g(t)$) be the characteristic (resp. minimal) polynomial of T. Let K be a field containing F. Show that if $\lambda \in K$ and $f(\lambda) = 0$, then $g(\lambda) = 0$. (Suggestion: First solve the problem assuming $K = F$. To prove the general statement, use the following trick (called *extending the scalars*): by taking a basis β of V, the problem can be equivalently expressed in terms of the matrix $A = [T]_{\beta} \in M_{n\times n}(F)$. Now the nice thing is that the characteristic polynomial of A is the same whether A is considered as an element of $M_{n\times n}(K)$ or $M_{n\times n}(F)$. Moreover, if the minimal polynomial of A as an element of $M_{n\times n}(K)$ is $h(t) \in K[t]$, then $h(t) \mid g(t)$ (why?).