MAT247 Algebra II

Assignment 6

Due Saturday March 9 at 11:59 pm
(to be submitted on Crowdmark)

Please write your solutions neatly and clearly. Note that due to time limitations, some questions may not be graded.

1. Let F be a field and $f(t), g(t), h(t) \in F[t]$. Suppose $f(t)$ and $g(t)$ are relatively prime.

(a) Show that if $f(t) \mid g(t)h(t)$, then $f(t) \mid h(t)$.

(b) Show that if $f(t)$ and $g(t)$ both divide $h(t)$, then $f(t)g(t) \mid h(t)$.

(c) Show that for every positive integers m and n, $f(t)^m$ and $g(t)^n$ are relatively prime. (Note: If two polynomials have a common divisor $q(x)$ of positive degree, then they have an irreducible common divisor, because $q(x)$ has an irreducible divisor.)

2. Let V be a nonzero finite-dimensional vector space over $\mathbb{C}$. Denote the identity map on V by I. Let T a linear operator on V such that $T^k = I$ for some positive integer k. Show that T is diagonalizable. (Suggestion: Use Theorem 7.16 (we'll prove it in class on Tuesday).)

3. Let V be a nonzero finite-dimensional vector space and T a diagonalizable linear operator on T. Let W be a T-invariant subspace of V. Show that T_W (the restriction of T to W) is diagonalizable. (Suggestion: Use Theorem 7.16. Does the minimal polynomial of T_W divide the minimal polynomial of T?)

4. Let V be a nonzero finite-dimensional vector space. Let $\mathcal{S}$ be a collection of diagonalizable linear operators on V such that any two maps in $\mathcal{S}$ commute with each other. Show that the maps in $\mathcal{S}$ can be simultaneously diagonalized. That is, show that there exists a basis β of V such that for every $T \in \mathcal{S}$, the matrix $[T]_\beta$ is diagonal. (Suggestion: Argue by induction on the dimension of V. In the induction step, consider two cases: (i) if every $T \in \mathcal{S}$ has only one eigenvalue, and (ii) if there exists $T \in \mathcal{S}$ which has at least two eigenvalues.)

5. Let F be a field and $A \in M_{n\times n}(F)$. By the minimal polynomial of A over F we mean the minimal polynomial of $L_A : F^n \to F^n$. Equivalently, the minimal polynomial of A over F is the unique monic polynomial $f(t) \in F[t]$ satisfying the following properties: (i) $f(A) = 0$, and (ii) if $g(t) \in F[t]$ is any nonzero polynomial such that $g(A) = 0$, then $\deg(f(t)) \leq \deg(g(t))$.

(a) Show that the degree of the minimal polynomial of A over F is equal to the smallest integer k such that there exists a nonzero vector $(c_0, \ldots, c_k) \in F^{k+1}$ such that

$$c_0 I + c_1 A + c_2 A^2 + \cdots + c_k A^k = 0.$$

(b) Let K be a field that contains F (as a subfield). Show that the minimal polynomial of A over F is the same as its minimal polynomial over K. (Hint: Let $B \in M_{\ell\times m}(F)$. If the equation $Bx = 0$ has a solution in K^m, then does it also have a solution in F^m?)

6. Suppose $A \in M_{5\times 5}(\mathbb{Q})$ has characteristic polynomial $f(t) = (t+1)^4(t-2)$. Let $g(t)$ be the minimal polynomial of A.

(a) List all possibilities for $g(t)$. What is the Jordan canonical form of A in each case? (List all possible Jordan canonical forms if there is more than one.)

(b) Suppose $g(t) = (t+1)^2(t-2)$ and that moreover $\dim(N(A+I)) = 2$. What is the Jordan canonical form of A?

(Suggestion: See exercise 13 of 7.3.)

Practice Problems: The following problems are for your practice. They are not to be handed in for grading.

From the textbook: exercises # 1, 2, 3, 5, 11, 12 of 7.3