MAT247 Algebra II

Assignment 7

Due Saturday March 16 at 11:59 pm
(to be submitted on Crowdmark)

Please write your solutions neatly and clearly. Note that due to time limitations, some questions may not be graded. Questions 1 and 2 below are exercises on how to use the existence and uniqueness of the rational canonical form. Questions 3 and 4 give the main components of the proof of uniqueness of the rational canonical form. Question 5 introduces a notion of direct sum, which is different from (but closely related to) the one we have been using so far in the course.

Notation. In what follows we shall use the following notation: for a polynomial f, we denote the companion matrix of f by $C(f)$. Given a scalar λ, we denote the $n \times n$ Jordan block corresponding to λ by $J_{\lambda,n}$.

1. Let F be a field.

(a) Let $\phi_1, \phi_2, \dots, \phi_k$ be distinct monic irreducible polynomials in $F[t]$. For each i, let $p_{i,j}$ $(1 \le j \le e_i)$ be integers with

$$p_{i,e_i} \ge p_{i,e_i-1} \ge \cdots \ge p_{i,1} \ge 1.$$

Let R be the following block diagonal matrix:

$$\begin{pmatrix} C(\phi_1^{p_{1,e_1}}) & & & & & & & & \\ & \ddots & & & & & & & \\ & & C(\phi_1^{p_{1,1}}) & & & & & & \\ & & & C(\phi_2^{p_{2,e_2}}) & & & & & \\ & & & & \ddots & & & & \\ & & & & & C(\phi_2^{p_{2,1}}) & & & \\ & & & & & & \ddots & & \\ & & & & & & & C(\phi_k^{p_{k,e_k}}) & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & C(\phi_k^{p_{k,1}}) \end{pmatrix}.$$

Show that the minimal polynomial of R is $\prod_{i=1}^{k} \phi_i^{p_{i,e_i}}$. (Remember that the minimal polynomial of the companion matrix of a monic polynomial f is f itself. Also note that if A is a block diagonal matrix with diagonal blocks $A_1, A_2, \dots$ and $g \in F[t]$, then $g(A)$ is the block diagonal matrix with diagonal blocks $g(A_1), g(A_2), \dots$.)

(b) Let ϕ_1 and ϕ_2 be two distinct monic irreducible elements of $F[t]$. Consider the set $\mathcal{M}$ of all matrices A with entries in F whose characteristic polynomial is $\phi_1^4\phi_2^2$. Then similarity is an equivalence relation on $\mathcal{M}$. Find the number of equivalence classes of this relation, give a representative for each class, and give the minimal polynomial of the matrices in each class.

2. Consider the polynomials $\phi = t^2 - 2$, $\phi^+ = t - \sqrt{2}$, and $\phi^- = t + \sqrt{2}$ in $\mathbb{R}[t]$. Let A be a matrix with entries in $\mathbb{Q}$.

(a) Suppose the characteristic polynomial of A is ϕ^2. Find all possibilities for the rational canonical form of A over $\mathbb{Q}$.

(b) With A as in part (a), find all possibilities for the rational canonical form of A over $\mathbb{R}$. (Keep in mind that the entries of A are in $\mathbb{Q}$.)

(c) Now suppose the characteristic polynomial of A is ϕ^4. Again, find all possibilities for the rational canonical form of A over $\mathbb{Q}$ and over $\mathbb{R}$.

(d) Let F be a subfield of K. For $1 \le i \le k$, let ϕ_i be distinct monic irreducible polynomials in $F[t]$. Suppose each ϕ_i factors as $\prod_j \psi_{i,j}$ in $K[t]$, where the $\psi_{i,j}$ are distinct, monic, and irreducible in $K[t]$. Let A be a matrix with entries in F whose characteristic polynomial is $p_A = \pm \prod_i \phi_i^{n_i}$. Formulate a conjecture that describes how the rational canonical forms of A over F and K are related. You don't need to prove your conjecture.

3. (a) Let T be a linear operator on a finite-dimensional vector space V over F. Let $f, g \in F[t]$ be relatively prime. Show that the restriction of $f(T)$ to $\ker(g(T))$ is injective. (Note: The statement is equivalent to saying that $\ker(f(T)) \cap \ker(g(T)) = 0$, which we proved in class a few lectures ago. You should rewrite the proof.)

(b) Deduce that if ϕ and ψ are distinct monic irreducible polynomials in $F[t]$, and A is the companion matrix of ψ^m, then $\phi(A)$ is invertible.

4. Suppose T is a linear operator on a finite-dimensional vector space V over a field F. Suppose moreover that V is a T-cyclic subspace of itself, and that the characteristic polynomial of T is $\pm\phi^m$, where ϕ is a monic irreducible polynomial in $F[t]$. Let $d = \deg(\phi)$. Let $v \in V$ be a vector such that V is the T-cyclic subspace generated by v.

(a) Show that the set

$$I = \{\phi(T)^{m-1}(v),\ \phi(T)^{m-1}(T(v)),\ \phi(T)^{m-1}(T^2(v)), \dots,\ \phi(T)^{m-1}(T^{d-1}(v))\}$$

is linearly independent. (Hint: What is the minimal polynomial of T?)

(b) By Cayley-Hamilton, $\phi(T)^m = 0$, so that $\phi(T)$ is a nilpotent map. Show that the Jordan canonical form of $\phi(T)$ has the form

$$\begin{pmatrix} J_{0,m} & & & \\ & J_{0,m} & & \\ & & \ddots & \\ & & & J_{0,m} \end{pmatrix},$$

where there are d Jordan blocks in the matrix. In other words, show that the dot diagram for the eigenvalue zero of $\phi(T)$ consists of d columns of length m. (Hint: For each $w_i := \phi(T)^{m-1}(T^i(v))$ $(0 \le i < d)$, form a cycle of length m of generalized eigenvectors of $\phi(T)$ with initial vector w_i. Then use Theorem 7.6.)

(c) Let A be the companion matrix of ϕ^m (with ϕ as above: a monic irreducible element of $F[t]$ of degree d). Deduce from part (b) that for each $1 \le r \le m$, the matrix $\phi^r(A)$ has nullity rd.

Remark: From Problems 3(b) and 4 one can deduce uniqueness of rational canonical form. In fact, one can use them to show the following stronger statement: if R and R' are block diagonal matrices in $M_{n\times n}(F)$ with diagonal blocks that are companion matrices of powers of irreducible polynomials (that is, if they are matrices in rational canonical form), then, unless R and R' are obtained from each other by a permutation of the diagonal blocks, they are not similar over any field extension of F. (In short, this is because the nullities of $\phi^i(R)$ and $\phi^i(R')$ won't be the same for some r and some irreducible polynomial $\phi \in F[t]$.) This statement together with existence of rational canonical form can be used to show that if two matrices $A, B \in M_{n\times n}(F)$ are similar over a field extension of F, then A and B are already similar over F.

5. So far in MAT240 and MAT247, you have seen the notion of direct sum of a collection of subspaces of a given vector space. There is another notion of direct sum, which we introduce in this problem. Let F be a field and V_i $(1 \le i \le k)$ vector spaces over F. Consider the cartesian product

$$V_1 \times \cdots \times V_k := \{(v_1, \dots, v_k) \;:\; v_i \in V_i \text{ for each } 1 \le i \le k\}.$$

Equip this set with component-wise addition and scalar multiplication. That is, define

$$(v_1, \dots, v_k) + (w_1, \dots, w_k) := (v_1 + w_1, \dots, v_k + w_k)$$

and

$$c(v_1, \dots, v_k) := (cv_1, \dots, cv_k) \qquad (c \in F).$$

(In more compact notation, $(v_i)_{1\le i\le k} + (w_i)_{1\le i\le k} = (v_i + w_i)_{1\le i\le k}$ and $c(v_i)_{1\le i\le k} = (cv_i)_{1\le i\le k}$.) Then you can easily check that $V_1 \times \cdots \times V_k$ together with the operations defined above is a vector space. This vector space is called the direct sum of the V_i, and is denoted by $V_1 \oplus \cdots \oplus V_k$, or $\bigoplus_{i=1}^{k} V_i$.

For each $1 \le i \le k$, one has a natural injection

$$\iota_i : V_i \to V_1 \oplus \cdots \oplus V_k$$

sending $v \in V_i$ to the tuple with v in its i-th entry and zeros elsewhere. One also has a natural surjection (called the projection to the i-th component)

$$\pi_i : V_1 \oplus \cdots \oplus V_k \to V_i \qquad (v_1, \dots, v_k) \mapsto v_i.$$

(a) Show that if β_i is a basis of V_i, then $\bigcup_{i=1}^{k} \iota_i(\beta_i)$ is a basis of $\bigoplus_{i=1}^{k} V_i$. Conclude that if the V_i are finite-dimensional, then $\dim(\bigoplus_{i=1}^{k} V_i) = \sum_{i=1}^{k} \dim(V_i)$.

(b) Let W be any vector space (over the same field F). Show that a map $T : W \to \bigoplus_{i=1}^{k} V_i$ is linear if and only if the component maps $\pi_i \circ T : W \to V_i$ are linear.

(c) The goal of this part is to relate the notion of direct sum introduced here to the one we had seen earlier. Suppose $V_1, \dots, V_k$ are all subspaces of a vector space V. Let $\bigoplus_{i=1}^{k} V_i$ be the direct sum of the V_i, as introduced here. Then we have a natural map

$$\alpha : \bigoplus_{i=1}^{k} V_i \to V \qquad (v_1, \dots, v_k) \mapsto \sum_{i=1}^{k} v_i.$$

Note that the image of α is the subspace $\sum_{i=1}^{k} V_i$. Show that the sum of the subspaces V_i is direct (in the sense we had earlier) if and only if the map α is injective. (Thus when the sum of the subspaces V_i is direct, we have the above distinguished isomorphism between the two notions of direct sums for the subspaces V_i. Sometimes people use the term *internal* direct sum for the earlier notion of direct sum, in contrast to the notion defined here being *external*.

Practice Problems: The following problems are for your practice. They are not to be handed in for grading.

From the textbook: exercises # 1, 8, 9, 10, 12, 15, 16 of 7.3; exercises # 1, 3a-c, 4, 5, 8 of 7.4

Extra problems:

1. true or false: (a) If T is a linear operator on a finite-dimensional vector space, then the minimal and characteristic polynomials of T have the same irreducible factors.

(b) If ϕ is an irreducible factor of the characteristic polynomial of a linear operator T on a finite-dimensional vector space, then $\phi(T)$ is not injective.

2. Let F be a field and $\lambda \in F$. Show that $J_{\lambda,n}$ (the $n \times n$ Jordan block corresponding to λ) is similar to $C((t-\lambda)^n)$ (i.e. the companion matrix of $(t-\lambda)^n$). Deduce that if the characteristic polynomial of an operator splits, then the rational canonical of the map is obtained from its Jordan canonical form by simply replacing each Jordan block $J_{\lambda,n}$ by $C((t-\lambda)^n)$.

3. Let T be a linear operator on a finite-dimensional vector space V over a field F. Let $v \in V$. We call a polynomial $f \in F[t]$ a T-annihilator of v if (i) f is monic, (ii) $f(T)(v) = 0$, and (iii) if $g \in F[t]$ is any nonzero polynomial such that $g(T)(v) = 0$, then $\deg(f) \leq \deg(g)$.

(a) Show that every $v \in V$ has a unique T-annihilator.
(b) Show that if $g \in F[t]$ is any polynomial such that $g(T)(v) = 0$, then the T-annihilator of v divides g.
(c) true or false: If W is the T-cyclic subspace generated by v, then the minimal polynomial of T_W is the same as the T-annihilator of v.

4. Let ϕ be an irreducible factor of the characteristic polynomial of a linear operator T on a vector space V. Let m be the exponent of ϕ in the minimal polynomial of T. Show that for every $k \geq m$, $\ker(\phi(T)^k) = \ker(\phi(T)^m)$. Conclude that

$$K_\phi := \{v \in V : \phi(T)^k(v) = 0 \text{ for some positive integer } k\}$$

is equals to $\ker(\phi(T)^m$. Do not use anything that hasn't been proven in class. (Hint: Use 3(b) of the practice list in Assignment 4.)

5. Let $\mathcal{A}$ be the set of all elements of $M_{8\times 8}(\mathbb{C})$ with characteristic polynomial $f(t) = (t^2+1)^4$. Let $\mathcal{A}' = \mathcal{A} \cap M_{8\times 8}(\mathbb{R})$. Given $A, B \in M_{8\times 8}(\mathbb{C})$, write $A \sim_{\mathbb{R}} B$ (resp. $A \sim_{\mathbb{C}} B$) if there exists a matrix P in $M_{8\times 8}(\mathbb{R})$ (resp. $M_{8\times 8}(\mathbb{C})$) such that $A = PBP^{-1}$. Note that $\sim_{\mathbb{R}}$ and $\sim_{\mathbb{C}}$ are equivalence relations on both $\mathcal{A}$ and $\mathcal{A}'$.

(a) Give a complete set of representatives for the equivalences classes of $\sim_{\mathbb{R}}$ on $\mathcal{A}'$ (that is, give exactly one representative from each equivalence class). Give the minimal polynomial of each equivalence class.
(b) Give a complete set of representatives for the equivalences classes of $\sim_{\mathbb{C}}$ on $\mathcal{A}'$.
(c) Give a complete set of representatives for the equivalences classes of $\sim_{\mathbb{C}}$ on $\mathcal{A}$. Give the minimal polynomial of each equivalence class.
(d) Which equivalence classes of $\mathcal{A}$ with respect to $\sim_{\mathbb{C}}$ do not contain any matrices with real entries?

6. Let V_i $(1 \leq i \leq k)$ be vector spaces. For each j, let $\iota_j : V_j \to \bigoplus_{i=1}^{k} V_i$ and $\pi_j : \bigoplus_{i=1}^{k} V_i \to V_j$ be the j-th natural embedding and projection. Prove that $\bigoplus_{i=1}^{k} V_i$ satisfies the following *universal properties*:

(a) Given any vector space W and linear maps $T_i : W \to V_i$, there exists a unique linear map $T : W \to \bigoplus_{i=1}^{k} V_i$ such that $T_i = \pi_i \circ T$ for each i.

(b) Given any vector space W and linear maps $T_i : V_i \to W$, there exists a unique linear map $T : \bigoplus_{i=1}^{k} V_i \to W$ such that $T_i = T \circ \iota_i$ for each i.

7. Let V_i ($1 \le i \le k$) and W be vector spaces. Construct isomorphisms

$$\mathcal{L}(\bigoplus_{i=1}^{k} V_i, W) \longrightarrow \bigoplus_{i=1}^{k} \mathcal{L}(V_i, W)$$

and

$$\mathcal{L}(W, \bigoplus_{i=1}^{k} V_i) \longrightarrow \bigoplus_{i=1}^{k} \mathcal{L}(W, V_i).$$

Your isomorphisms should be natural (or canonical), in the sense that they should not depend on any choices (for example, of bases).† (Suggestion: previous problem.)

8. Here is a more general version of the bonus problem on the regular sitting of the midterm. Suppose V is a finite-dimensional vector space over F. Let $f(t), g(t) \in F[t]$ are relatively prime and $f(T)g(T) = 0$. We will then have a decomposition

$$V = \ker(f(T)) \oplus \ker(g(T))$$

(why?). Let $\pi : V \to V$ be the natural projection onto $\ker(f(T))$ relative to the decomposition above. Show that π is a polynomial in T.

9. Let V be an n-dimensional vector space. Let $T : V \to V$ be a linear operator such that $T^2 = T$. Show that if $\dim(\ker(T)) = d$, then the characteristic polynomial of T is $t^d(1-t)^{n-d}$.

†There is a more precise description of what it means for an isomorphism to be natural or canonical, which is beyond the scope of our course.