MAT247 Algebra II

Assignment 8

Solutions

1. (equality cases of Cauchy-Schwarz and triangle inequalities) Let V be an inner product space and $x, y \in V$.

(a) Show that
$$|\langle x, y\rangle| = \|x\| \cdot \|y\|$$
if and only if one of x or y is a scalar multiple of the other. (Suggestion: Go over the proof of Cauchy-Schwarz.)

(b) Show that
$$\|x + y\| = \|x\| + \|y\|$$
if and only if one of x or y is equal to a non-negative real number times the other. (Suggestion: Go over the proof of triangle inequality!)

Solution: (a) The "if" implication is clear and we leave it to the reader. Suppose $|\langle x, y\rangle| = \|x\| \cdot \|y\|$. If y is zero, then it is a scalar multiple of x and we are done. Suppose $y \neq 0$. For any $c \in F$, we have
$$\|x - cy\|^2 = \langle x - cy, x - cy\rangle = \|x\|^2 + |c|^2\|y\|^2 - c\langle y, x\rangle - \bar{c}\langle x, y\rangle.$$
Taking $c = \frac{\langle x,y\rangle}{\|y\|^2}$, this simplifies to
$$\|x - cy\|^2 = \frac{\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2}{\|y\|^2} = 0.$$
Thus $x = cy$.

(b) $\Rightarrow$: We may assume $y \neq 0$. We have
$$\|x + y\|^2 = \langle x + y, x + y\rangle^2 \overset{\text{why}}{=} \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle.$$
We also have
$$(\|x\| + \|y\|)^2 = \|x\|^2 + \|y\|^2 + 2\|x\|\ \|y\|.$$
Suppose $\|x + y\|^2 = (\|x\| + \|y\|)^2$. Then
$$\langle x, y\rangle + \langle y, x\rangle = 2\|x\|\ \|y\| \geq 2|\langle x, y\rangle|, \tag{1}$$
by Cauchy-Schwarz. Since $\langle y, x\rangle = \overline{\langle x, y\rangle}$, we get
$$\mathrm{Re}(\langle x, y\rangle) \geq |\langle x, y\rangle|,$$
where Re stands for the real part. It follows that $\langle x, y\rangle$ is real and nonnegative. Eq. (1) now gives
$$2\langle x, y\rangle| = 2\|x\|\ \|y\| \geq 2\langle x, y\rangle,$$
so that
$$\|x\|\ \|y\| = \langle x, y\rangle.$$
Thus by (a), there is $c \in \mathbb{C}$ such that $x = cy$. We have
$$0 \leq \langle x, y\rangle = c\langle y, y\rangle.$$
It follows that c is real and nonnegative, as desired.

$\Leftarrow$: Suppose without loss that $x = cy$ for some $c \geq 0$. Then

$$\|x + y\| = \|(c+1)y\| = |(c+1)|\|y\| = (c+1)\|y\|.$$

On the other hand,

$$\|x\| + \|y\| = \|cy\| + \|y\| = (|c|+1)\|y\| = (c+1)\|y\|.$$

2. Let V be an inner product space and $x, y \in V$.

(a) Prove the *parallelogram law*:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Specializing to the case of $\mathbb{R}^2$ with its standard inner product, what does this identity say about a parallelogram on the plane?

(b) Show that if $\langle x, y\rangle = 0$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Solution: We leave this to the reader.

3. Let $F = \mathbb{R}$ or $\mathbb{C}$ and V be an inner product space over F. For each $y \in V$, we have a map

$$\langle -, y\rangle : V \to F \quad x \mapsto \langle x, y\rangle.$$

Linearity of $\langle\ ,\ \rangle$ in the first component implies that $\langle -, y\rangle$ is linear, so that it belongs to $V^\vee$.

(a) Show that the function

$$\alpha : V \to V^\vee$$

defined by $\alpha(y) = \langle -, y\rangle$ is injective, and that moreover it is linear when $F = \mathbb{R}$.

(b) Conclude that if $F = \mathbb{R}$ and V is finite-dimensional, then α is an isomorphism. (Remark: In general, there is no natural (or distinguished) isomorphism between a finite-dimensional vector space and its dual. But we see here that if a finite-dimensional vector space over $\mathbb{R}$ is equipped with an inner product, then the inner product gives rise to a natural isomorphism between the vector space and its dual.)

(c) Suppose $F = \mathbb{R}$ or $\mathbb{C}$. Let V be finite-dimensional and $\{v_1, \ldots, v_n\}$ an orthonormal basis of V. Let $f \in V^\vee$. Show that $f = \sum_{i=1}^{n} f(v_i)\langle -, v_i\rangle$.

Solution: (a) By antilinearily of an inner product in the second factor, α is an $\mathbb{R}$-linear map (even when $F = \mathbb{C}$). For injectivity, suppose $y \in \ker(\alpha)$. Then $\langle x, y\rangle = 0$ for all x, and in particular, $\langle y, y\rangle = 0$. Thus $y = 0$ and $\ker(\alpha) = 0$.

(b) The map α is an injective linear map and V and $V^\vee$ are finite dimensional vector spaces of the same dimension, so α is an isomorphism. (Note that even if $F = \mathbb{C}$, by the same reasoning, α is an isomorphism between the underlying real vector spaces of V and $V^\vee$.)

(c) Both f and $g = \sum_{i=1}^{n} f(v_i)\langle -, v_i\rangle$ are linear maps $V \to F$. To prove they are equal, it is enough to check that they agree on a basis of V, say on $\{v_1, \ldots, v_n\}$. For each v_j,

$$g(v_j) = \sum_{i=1}^{n} f(v_i)\langle v_j, v_i\rangle = f(v_j),$$

where the last equality is by orthonormality of $\{v_1, \ldots, v_n\}$.

4. Let V be an inner product space (real or complex, possibly infinite-dimensional). Let $\{v_1, \ldots, v_n\}$ be an orthonormal set of vectors.

(a) Show that

$$\|\sum_{i=1}^{n} c_i v_i\|^2 = \sum_{i=1}^{n} |c_i|^2.$$

(b) Show that for every $x \in V$,

$$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i\rangle|^2,$$

with equality holding if and only if $x \in span\{v_1, \ldots, v_n\}$.

(c) Consider the space $C_{\mathbb{C}}[0,1]$ of continuous complex-valued functions on the interval $[0,1]$, equipped with an inner product $\langle\ ,\ \rangle$ defined by

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}dt.$$

For any integer k, let $f_k : [0,1] \to \mathbb{C}$ be the function defined by $f_k(t) = e^{2k\pi i t}$. Show that $\{f_k : k \in \mathbb{Z}\}$ is an orthonormal set. Conclude that for any $g \in V$ and any positive integer n,

$$\|g\|^2 \geq \sum_{|k|\leq n} |\langle g, f_k\rangle|^2.$$

(d) Show that

$$\frac{\pi^2}{6} \geq \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

(Suggestion: Take g to be the function defined by $g(t) = t$ and apply the inequality of (c). Note: This is an example in your textbook, but you should do it yourself.)

Solution: (a) We have

$$\|\sum_{i=1}^{n} c_i v_i\|^2 = \langle\sum_{i=1}^{n} c_i v_i, \sum_{i=1}^{n} c_i v_i\rangle = \sum_{i,j} c_i\overline{c_j}\langle v_i, v_j\rangle = \sum_i |c_i|^2,$$

by orthonormality of the v_i.

(b) Let $U = span\{v_1, \ldots, v_n\}$. Since U is finite-dimensional, we have $V = U \oplus U^\perp$. Given $x \in V$, writing $x = y + z$ with $y \in U$ and $z \in U^\perp$, the vector y is given by the formula

$$y = \sum_{i=1}^{n} \langle x, v_i\rangle v_i.$$

By the Pythagorean theorem (problem 2b),

$$\|x\|^2 = \|z\|^2 + \|y\|^2 \overset{\text{part (a)}}{=} \|z\|^2 + \sum_i |\langle x, v_i\rangle|^2.$$

This proves the desired inequality. Moreover, we have

$$\|x\|^2 = \sum_{i=1}^{n} |\langle x, v_i\rangle|^2$$

if and only if $z = 0$, which is equivalent to $x \in U$.

For (c) and (d) See Example 9 of 6.1 and Example 7 of 6.2.

5. Consider the complex vector space $P_4(\mathbb{C})$ of polynomials of degree at most 4 with coefficients in $\mathbb{C}$, equipped with the inner product $\langle\ ,\ \rangle$ defined by

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

(a) Find an orthogonal basis of the subspace $P_1(\mathbb{C}) = span\{1, x\}$.
(b) Find the element of $P_1(\mathbb{C})$ that is closest to x^2.
(c) Find an orthogonal basis for $P_1(\mathbb{C})^\perp$ (= the orthogonal complement of $P_1(\mathbb{C})$).

Solution: (a) We apply the Gram-Schmidt process to the basis $\{1, x\}$ of $P_1(\mathbb{C})$:

$$x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} \cdot 1 = x - \frac{1}{2}.$$

$\{1, x - \frac{1}{2}\}$ is an orthogonal basis of $P_1(\mathbb{C})$.

(b) The closest vector to x^2 in $P_1(\mathbb{C})$ is the orthogonal projection of x^2 to $P_1(\mathbb{C})$, which is

$$\frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} \cdot 1 + \frac{\langle x^2, x - 1/2\rangle}{\langle x - 1/2, x - 1/2\rangle}(x - 1/2) = x - 1/6.$$

(c) Extend the basis $\{1, x - 1/2\}$ of $P_1(\mathbb{C})$ to a basis, say, $\beta = \{1, x - 1/2, x^2, x^3, x^4\}$ of $P_4(\mathbb{C})$. Applying Gram-Schmidt to β we get a basis $\gamma = \{1, x - 1/2, x^2 - x + 1/6, v, v'\}$ (we leave it to the reader to find v, v'). Then $\{x^2 - x + 1/6, v, v'\}$ is an orthogonal basis of $P_1(\mathbb{C})^\perp$.