MAT247 Algebra II

Assignment 8

Due Saturday March 23 at 11:59 pm
(to be submitted on Crowdmark)

Please write your solutions neatly and clearly. Note that due to time limitations, some questions may not be graded.

1. (equality cases of Cauchy-Schwarz and triangle inequalities) Let V be an inner product space and $x, y \in V$.

(a) Show that
$$|\langle x, y\rangle| = \|x\| \cdot \|y\|$$
if and only if one of x or y is a scalar multiple of the other. (Suggestion: Go over the proof of Cauchy-Schwarz.)

(b) Show that
$$\|x + y\| = \|x\| + \|y\|$$
if and only if one of x or y is equal to a non-negative real number times the other. (Suggestion: Go over the proof of triangle inequality!)

2. Let V be an inner product space and $x, y \in V$.

(a) Prove the *parallelogram law*:
$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$
Specializing to the case of $\mathbb{R}^2$ with its standard inner product, what does this identity say about a parallelogram on the plane?

(b) Show that if $\langle x, y\rangle = 0$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

3. Let $F = \mathbb{R}$ or $\mathbb{C}$ and V be an inner product space over F. For each $y \in V$, we have a map
$$\langle -, y\rangle : V \to F \quad x \mapsto \langle x, y\rangle.$$
Linearity of $\langle\ ,\ \rangle$ in the first component implies that $\langle -, y\rangle$ is linear, so that it belongs to $V^\vee$.

(a) Show that the function
$$\alpha : V \to V^\vee$$
defined by $\alpha(y) = \langle -, y\rangle$ is injective, and that moreover it is linear when $F = \mathbb{R}$.

(b) Conclude that if $F = \mathbb{R}$ and V is finite-dimensional, then α is an isomorphism. (Remark: In general, there is no natural (or distinguished) isomorphism between a finite-dimensional vector space and its dual. But we see here that if a finite-dimensional vector space over $\mathbb{R}$ is equipped with an inner product, then the inner product gives rise to a natural isomorphism between the vector space and its dual.)

(c) Suppose $F = \mathbb{R}$ or $\mathbb{C}$. Let V be finite-dimensional and $\{v_1, \ldots, v_n\}$ an orthonormal basis of V. Let $f \in V^\vee$. Show that $f = \sum_{i=1}^{n} f(v_i)\langle -, v_i\rangle$.

4. Let V be an inner product space (real or complex, possibly infinite-dimensional). Let $\{v_1, \ldots, v_n\}$ be an orthonormal set of vectors.

(a) Show that
$$\|\sum_{i=1}^{n} c_i v_i\|^2 = \sum_{i=1}^{n} |c_i|^2.$$

(b) Show that for every $x \in V$,

$$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i \rangle|^2,$$

with equality holding if and only if $x \in \operatorname{span}\{v_1, \ldots, v_n\}$.

(c) Consider the space $C_{\mathbb{C}}[0,1]$ of continuous complex-valued functions on the interval $[0,1]$, equipped with an inner product $\langle \, , \, \rangle$ defined by

$$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}dt.$$

For any integer k, let $f_k : [0,1] \to \mathbb{C}$ be the function defined by $f_k(t) = e^{2k\pi i t}$. Show that $\{f_k : k \in \mathbb{Z}\}$ is an orthonormal set. Conclude that for any $g \in V$ and any positive integer n,

$$\|g\|^2 \geq \sum_{|k| \leq n} |\langle g, f_k \rangle|^2.$$

(d) Show that

$$\frac{\pi^2}{6} \geq \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

(Suggestion: Take g to be the function defined by $g(t) = t$ and apply the inequality of (c). Note: This is an example in your textbook, but you should do it yourself.)

5. Consider the complex vector space $P_4(\mathbb{C})$ of polynomials of degree at most 4 with coefficients in $\mathbb{C}$, equipped with the inner product $\langle \, , \, \rangle$ defined by

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

(a) Find an orthogonal basis of the subspace $P_1(\mathbb{C}) = \operatorname{span}\{1, x\}$.
(b) Find the element of $P_1(\mathbb{C})$ that is closest to x^2.
(c) Find an orthogonal basis for $P_1(\mathbb{C})^{\perp}$ (= the orthogonal complement of $P_1(\mathbb{C})$).

Practice Problems: The following problems are for your practice. They are not to be handed in for grading.

From the textbook: end of section 6.1 exercises, in particular, exercises # 1, 4, 8, 9, 13, 16-19, 24 (the definition of a norm is given just before the exercise), 25 (this gives an example of a norm that doesn't arise from an inner product), 26-30. (Exercises # 27 and 30 give a criterion for a norm to come from an inner product.)

Extra problems: To be added.