MAT247 Algebra II

Assignment 9

Solutions

1. Let $n \geq 1$ and consider $P_n(\mathbb{R})$ (= space of polynomial functions of degree at most n with real coefficients). Let $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the differentiation map, defined by $T(f) = f'$.

(a) Show that T is not normal with respect to any inner product on $P_n(\mathbb{R})$.

(b) Let $n = 1$ and consider $P_1(\mathbb{R})$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$

Find the adjoint map T^*.

Solution: (a) Suppose T is normal with respect to some inner product on $P_n(\mathbb{R})$. Since the characteristic polynomial of T splits over $\mathbb{R}$ (since T is nilpotent), it follows that T is self-adjoint, and hence in particular, diagonalizable. But this is absurd because T is nonzero and nilpotent.

(b) First find an orthonomal basis β of $P_1(\mathbb{R})$. Then use the formula $[T^*]_\beta = [T]_\beta^*$. The computations are left to the reader.

2. Let F be $\mathbb{R}$ or $\mathbb{C}$. Let V be the space of everywhere infinitely differentiable functions $f : \mathbb{R} \to F$ which satisfy $f(x+1) = f(x)$ for all x. Define $\langle\ ,\ \rangle$ on V by

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

(Because of 1-periodicity this is indeed an inner product.) Define $T : V \to V$ by $T(f) = f'$. Find T^*. Is T normal? Is it self-adjoint?

Solution: We claim that $T^* = -T$ (so that T is normal, but not self-adjoint). Indeed,

$$\langle T(f), g\rangle = \int_0^1 f'(x)\overline{g(x)}dx.$$

Integrating by parts,

$$\int_0^1 f'(x)\overline{g(x)}dx = f(x)\overline{g(x)}\ \Big|_{x=0}^{x=1} - \int_0^1 f(x)\overline{g(x)}'dx \overset{\text{periodicity}}{=} -\int_0^1 f(x)\overline{g(x)}'dx = \langle f, -g'\rangle = \langle f, -T(g)\rangle.$$

Thus $-T$ satisfies the defining property of the adjoint of T.

3. Let F be $\mathbb{R}$ or $\mathbb{C}$, and V a (possibly infinite-dimensional) inner product space over F. Let T be a normal operator on V.

(a) Show that $\|T(x)\| = \|T^*(x)\|$ for any $x \in V$.

(b) Show that if x is an eigenvector of T with eigenvalue λ, then x is an eigenvector of T^* with eigenvalue $\overline{\lambda}$.

(c) Show that if λ, λ' are distinct eigenvalues of T, $x \in E_\lambda$ and $x' \in E_{\lambda'}$, then $\langle x, x' \rangle = 0$.

Solution: See Theorem 6.15 of the textbook.

4. Let F be $\mathbb{R}$ or $\mathbb{C}$, and V a finite-dimensional inner product space over F. Let $T : V \to V$ be a linear operator. Show that T is normal if and only if there exists a polynomial $f(t) \in F[t]$ such that $T^* = f(T)$.

Solution: The implication $\Leftarrow$ is clear, since T commutes with any polynomial in T. Below we prove $\Rightarrow$.

First, let $F = \mathbb{C}$. Then by the spectral theorem, there exists an orthonormal basis $\beta = \{v_1, \dots, v_n\}$ of V consisting of eigenvectors of T. Let $T(v_i) = \lambda_i v_i$. Then $[T]_\beta$ is the diagonal matrix with entries $\lambda_1, \dots, \lambda_n$, and $[T^*]_\beta = [T]_\beta^*$ is the diagonal matrix with entries $\overline{\lambda_1}, \dots, \overline{\lambda_n}$. Let $f(t) \in \mathbb{C}[t]$ be such that $f(\lambda_i) = \overline{\lambda_i}$ for all i. (Such $f(t)$ can be constructed using Lagrange interpolation.) Then

$$[f(T)]_\beta = f([T]_\beta) = [T]_\beta^* = [T^*]_\beta,$$

so that $f(T) = T^*$.

Now let us consider the case $F = \mathbb{R}$. Let γ be an orthonormal basis of V and $A = [T]_\alpha$. Let $L_A : \mathbb{C}^n \to \mathbb{C}^n$ be left multiplication by A. Since T is normal, the matrix A is normal, and hence so is L_A (where $\mathbb{C}^n$ is considered with its standard inner product). By the complex case of the spectral theorem, there exist an orthonormal basis β of $\mathbb{C}^n$ consisting of eigenvectors of L_A (or of A). Let $\mu_1, \dots, \mu_k$ be the distinct eigenvalues of L_A. If $f(t) \in \mathbb{C}[t]$ is a polynomial such that $f(\mu_i) = \overline{\mu_i}$ for all i, then $f([L_A]_\beta) = [L_A]_\beta^*$, so that

$$[L_{f(A)}]_\beta = [f(L_A)]_\beta = f([L_A]_\beta) = [L_A]_\beta^* = [L_A^*]_\beta = [L_{A^*}]_\beta,$$

and thus $f(A) = A^*$, i.e. $f([T]_\alpha) = [T^*]_\alpha$. If $f(t) \in \mathbb{R}[t]$, then this gives $[f(T)]_\alpha = [T^*]_\alpha$, and hence $f(T) = T^*$, as desired. Thus we will be done if we prove that there exists $f(t) \in \mathbb{R}[t]$ such that $f(\mu_i) = \overline{\mu_i}$ for all $1 \le i \le k$. By Lagrange interpolation, there exists a polynomial $f(t) \in \mathbb{C}[t]$ of degree $\le k-1$ which satisfies $f(\mu_i) = \overline{\mu_i}$ for all i. We claim that this $f(t)$ actually has real coefficients. Indeed, we shall show that $\bar{f}(t) = f(t)$, where $\bar{f}(t)$ is the polynomial whose coefficients are the complex conjugates of those of $f(t)$. Since $f(t)$ and $\bar{f}(t)$ have degree at most $k-1$, it is enough to show that $f(t)$ and $\bar{f}(t)$ agree at the k points $\mu_1, \dots, \mu_k$. Since A is a matrix with real coefficients, the complex conjugate of each eigenvalue of A (or each root of the characteristic polynomial of A) is also an eigenvalue of A. Thus $\overline{\mu_i}$ is μ_j for some j, and hence $f(\overline{\mu_i}) = \mu_i$. We have

$$\bar{f}(\mu_i) = \overline{f(\overline{\mu_i})} = \overline{\mu_i} = f(\mu_i).$$

5. Let F be $\mathbb{R}$ or $\mathbb{C}$.

(a) Let $A \in M_{n \times n}(F)$. Define

$$\langle x, y \rangle_A = x^t A \bar{y}.$$

Show that $\langle \ , \ \rangle_A$ is an inner product on F^n if and only if A is a self-adjoint matrix all whose eigenvalues are positive. (Such A is called positive-definite.)

(b) Show that every inner product on F^n is of the form $\langle \ , \ \rangle_A$ for a unique self-adjoint $A \in M_{n \times n}(F)$ with positive eigenvalues.

Solution: (a) It is clear that $\langle \ , \ \rangle_A$ is linear in the first factor and antilinear in the second. We now show that A is self-adjoint if and only if $\langle \ , \ \rangle_A$ is conjugate-symmetric. Indeed, $\langle x, y \rangle_A =$

$x^t A\overline{y}$ and

$$\overline{\langle y, x\rangle_A} = \overline{y^t A \overline{x}} = y^* \overline{A} x = (y^* \overline{A} x)^t = x^t A^* \overline{y}.$$

Thus $\langle\ ,\ \rangle_A$ is conjugate-symmetric if and only if for every x and y,

$$x^t A \overline{y} = x^t A^* \overline{y}.$$

This is equivalent to $A = A^*$. (Taking $x = e_i$ and $y = e_j$, the equation above reads $A_{ij} = (A^*)_{ij}$.)

Now suppose that A is self-adjoint. Then the characteristic polynomial of A splits over $\mathbb{R}$. We show that $\langle\ ,\ \rangle_A$ is positive-definite if and only if all the eigenvalues of A are positive. By the spectral theorem, A is diagonalizable over F, and moreover there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of F^n consisting of eigenvectors of A (orthonormality with respect to the standard product). Let $Av_i = \lambda_i v_i$. Then the λ_i are real (since A is self-adjoint), and our goal is to show that $\langle\ ,\ \rangle_A$ is positive-definite if and only if all the λ_i are positive. Let $x \in F^n$. Write $\overline{x} = \sum_i a_i v_i$ for $a_i \in F$ $(1 \le i \le n)$. Then $x = \sum_i \overline{a_i}\, \overline{v_i}$, and

$$\begin{aligned}
\langle x, x\rangle_A = x^t A \overline{x} &= (\sum_i \overline{a_i} v_i^*) A (\sum_i a_i v_i) \\
&= \sum_{i,j} \overline{a_i} a_j v_i^* A v_j \\
&= \sum_{i,j} \overline{a_i} a_j \lambda_j v_i^* v_j \\
&= \sum_i \overline{a_i} a_i \lambda_i,
\end{aligned}$$

by orthonormality of the v_i. Thus

$$\langle x, x\rangle_A = \sum_i |a_i|^2 \lambda_i.$$

If all the λ_i are positive, then for every nonzero x,

$$\sum_i |a_i|^2 \lambda_i \ > \ 0.$$

On the other hand, if one of the λ_i, say λ_j , is ≤ 0, then $\langle \overline{v_j}, \overline{v_j}\rangle_A = \lambda_j \le 0$.

(b) Let $\langle\ ,\ \rangle$ be an inner product on F^n. Let $A \in M_{n\times n}(F)$ be the matrix with $A_{ij} = \langle e_i, e_j\rangle$, where the $\{e_1, \ldots, e_n\}$ is the standard basis of F^n. Then we can easily see that $\langle\ ,\ \rangle = \langle\ ,\ \rangle_A$. Indeed, since both $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_A$ are linear in the first factor and antilinear in the second, it is enough to have $\langle e_i, e_j\rangle = \langle e_i, e_j\rangle_A$ for all i, j. This property holds thanks to the definition of A. (Note that $\langle e_i, e_j\rangle_A$ is just the ij-entry of A.) Since $\langle\ ,\ \rangle$ is an inner product, by (a), A is positive-definite. This proves the existence assertion. For the uniqueness assertion, suppose $\langle x, y\rangle_A = \langle x, y\rangle_B$ for all x and y. Taking $x = e_i$ and $y = e_j$, we get $A_{ij} = B_{ij}$, so that $A = B$.

6. Let F be $\mathbb{R}$ or $\mathbb{C}$. Let V and W be inner product spaces over F. With abuse of notation, we denote both inner products (resp. norms) by $\langle\ ,\ \rangle$ (resp. $\|\ \|$). Let $T : V \to W$ be linear.

(a) Show that the following statements are equivalent:
   (i) T is norm-preserving, i.e. for every $x \in V$, we have $\|x\| = \|T(x)\|$.
   (ii) T is distance-preserving, i.e. for every $x, y \in V$, we have $\|x - y\| = \|T(x) - T(y)\|$.
   (iii) T preserves the inner product, i.e. for every $x, y \in V$, we have $\langle x, y\rangle = \langle T(x), T(y)\rangle$.

(b) A linear map $T : V \to W$ that satisfies the equivalent conditions of Part (a) is called a linear isometry. Show that a linear isometry is injective. Also show that the inverse of a bijective linear isometry is also a linear isometry. (A bijective linear isometry between inner product spaces is called an isomorphism of inner product spaces. The terms unitary when $F = \mathbb{C}$ and orthogonal when $F = \mathbb{R}$ are also used for such map. The latter two terms are especially used in the case where V and W are the same inner product space.)

(c) Suppose V and W are finite-dimensional and $\dim(V) \leq \dim(W)$. Construct a linear isometry $V \to W$. (In particular, if V is any n-dimensional inner product space over F, V is isomorphic as an inner product space to F^n with the standard inner product.) Hint: Start by taking orthonormal bases of V and W.

Solution: We did (a) and (b) in class. As for (c), let $\{v_1, \ldots, v_m\}$ (resp. $\{w_1, \ldots, w_n\}$) be an orthonormal basis of V (resp. W). Since $m \leq n$, we can define a linear map T by setting $T(v_i) = w_i$ for $1 \leq i \leq m$. Since T maps an orthonormal basis of V to an orthonormal set in W, it respects the inner products.