MAT247 Algebra II

Assignment 9

Due Friday April 5 at 11:59 pm
(to be submitted on Crowdmark)

Please write your solutions neatly and clearly. Note that due to time limitations, some questions may not be graded.

1. Let $n \geq 1$ and consider $P_n(\mathbb{R})$ (= space of polynomial functions of degree at most n with real coefficients). Let $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the differentiation map, defined by $T(f) = f'$.

(a) Show that T is not normal with respect to any inner product on $P_n(\mathbb{R})$.

(b) Let $n = 1$ and consider $P_1(\mathbb{R})$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

Find the adjoint map T^*.

2. Let F be $\mathbb{R}$ or $\mathbb{C}$. Let V be the space of everywhere infinitely differentiable functions $f : \mathbb{R} \to F$ which satisfy $f(x + 1) = f(x)$ for all x. Define $\langle \ , \ \rangle$ on V by

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

(Because of 1-periodicity this is indeed an inner product.) Define $T : V \to V$ by $T(f) = f'$. Find T^*. Is T normal? Is it self-adjoint?

3. Let F be $\mathbb{R}$ or $\mathbb{C}$, and V a (possibly infinite-dimensional) inner product space over F. Let T be a normal operator on V.

(a) Show that $\|T(x)\| = \|T^*(x)\|$ for any $x \in V$.

(b) Show that if x is an eigenvector of T with eigenvalue λ, then x is an eigenvector of T^* with eigenvalue $\overline{\lambda}$.

(c) Show that if λ, λ' are distinct eigenvalues of T, $x \in E_\lambda$ and $x' \in E_{\lambda'}$, then $\langle x, x' \rangle = 0$.

Note: This is Theorem 6.15 of the textbook, but you should prove the statements yourself.

4. Let F be $\mathbb{R}$ or $\mathbb{C}$, and V a finite-dimensional inner product space over F. Let $T : V \to V$ be a linear operator. Show that T is normal if and only if there exists a polynomial $f(t) \in F[t]$ such that $T^* = f(T)$.

5. Let F be $\mathbb{R}$ or $\mathbb{C}$.

(a) Let $A \in M_{n\times n}(F)$. Define

$$\langle x, y \rangle_A = x^t A \overline{y}.$$

Show that $\langle \ , \ \rangle_A$ is an inner product on F^n if and only if A is a self-adjoint matrix all whose eigenvalues are positive. (Such A is called positive-definite.)

(b) Show that every inner product on F^n is of the form $\langle \ , \ \rangle_A$ for a unique self-adjoint $A \in M_{n\times n}(F)$ with positive eigenvalues.

6. Let F be $\mathbb{R}$ or $\mathbb{C}$. Let V and W be inner product spaces over F. With abuse of notation, we denote both inner products (resp. norms) by $\langle \ , \ \rangle$ (resp. $\| \ \|$). Let $T : V \to W$ be linear.

(a) Show that the following statements are equivalent:

(i) T is norm-preserving, i.e. for every $x \in V$, we have $\|x\| = \|T(x)\|$.
(ii) T is distance-preserving, i.e. for every $x, y \in V$, we have $\|x - y\| = \|T(x) - T(y)\|$.
(iii) T preserves the inner product, i.e. for every $x, y \in V$, we have $\langle x, y \rangle = \langle T(x), T(y) \rangle$.

(b) A linear map $T : V \to W$ that satisfies the equivalent conditions of Part (a) is called a linear isometry. Show that a linear isometry is injective. Also show that the inverse of a bijective linear isometry is also a linear isometry. (A bijective linear isometry between inner product spaces is called an isomorphism of inner product spaces. The terms unitary when $F = \mathbb{C}$ and orthogonal when $F = \mathbb{R}$ are also used for such map. The latter two terms are especially used in the case where V and W are the same inner product space.)

(c) Suppose V and W are finite-dimensional and $\dim(V) \leq \dim(W)$. Construct a linear isometry $V \to W$. (In particular, if V is any n-dimensional inner product space over F, V is isomorphic as an inner product space to F^n with the standard inner product.) Hint: Start by taking orthonormal bases of V and W.

Practice Problems: The following problems are for your practice. They are not to be handed in for grading.

From the textbook: from 6.3: # 1, 2, 3, 6, 8, 9, 10, 13, 14, 15, 18, 19; from 6.4: # 1, 2, 4, 6, 7, 8, 9, 10, 13, 18, 21, 23; from 6.6: # 1, 4, 5, 6 (a linear operator T is called a projection if $T^2 = T$), 10

Extra problems:

1. Let T be a linear operator on a finite-dimensional inner product space. Show that the characteristic polynomial of T^* is the complex conjugate of the characteristic polynomial of T. (If $f(t) = \sum a_i t^i$, by definition, $\bar{f}(t) = \sum \overline{a_i} t^i$.)

2. Let $F = \mathbb{R}$ or $\mathbb{C}$ and V be an inner product space over F. Let T be a linear operator on V. Show that if T is normal (resp. self-adjoint), then so is $f(T)$ for any polynomial in $f(t) \in F[t]$ (resp. $f(t) \in \mathbb{R}[t]$).

3. Let $A \in M_{m\times n}(\mathbb{C})$.

(a) Let $x \in \mathbb{C}^n$. Show that if $A^*Ax = 0$, then $Ax = 0$. (Hint: Consider $\langle A^*Ax, x\rangle$, where $\langle\ ,\ \rangle$ is the standard inner product on $\mathbb{C}^n$.)

(b) Show that $\mathrm{rank}(A^*A) = \mathrm{rank}(A)$.

4. Let V be an n-dimensional complex vector space. Let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V. Define V' be the $\mathbb{R}$-span of β, i.e.

$$V' = \{\sum a_i v_i : a_i \in \mathbb{R}\}.$$

Show that any real inner product on V' extends uniquely to a complex inner product on V. Conclude that there is a one-to-one correspondence between the real inner products on V' and the complex inner products on V which are real-valued on V'.

5. Let V be a real or complex vector space. Let $V = U \oplus W$. Suppose $\langle\ ,\ \rangle_U$ and $\langle\ ,\ \rangle_W$ are inner products on U and W. Show that there exists a unique inner product on V whose restriction to U (resp. W) is $\langle\ ,\ \rangle_U$ (resp. $\langle\ ,\ \rangle_W$), and with respect to which U and W are orthogonal complements of one another.