MAT247 Midterm Solution

1. [10 points] Determine if each statement below is true or false. No explanation is necessary.

(a) If V is a finite-dimensional vector space over $\mathbb{C}$, then the dimension of V as a vector space over $\mathbb{R}$ is even.

true

(b) Any two matrices in $M_{3\times 3}(\mathbb{Q})$ with characteristic polynomials $t(1-t)(t+1)$ are similar.

true

(c) If T is a linear operator on a finite-dimensional vector space V, then for every T-invariant subspace U of V, there exists a T-invariant subspace W of V such that $V = U \oplus W$.

false

(d) If T is an invertible linear operator on a finite-dimensional vector space V over a field F, then there exists a polynomial $f(t) \in F[t]$ such that $T^{-1} = f(T)$.

true

(e) Suppose T is a linear operator on a finite-dimensional vector space V. If there exists a basis β of T such that $[T]_\beta$ is upper triangular, then there exists a basis γ of V such that $[T]_\gamma$ is a Jordan matrix.

true

2. [8 points] Let V be a vector space over $\mathbb{Q}$ of dimension 5 and T a linear operator on V with characteristic polynomial $f(t) = -t(t-1)^4$.

(a) Give all possible dot diagrams for eigenvalue $\lambda = 1$ of T.

(b) Suppose $\dim(\mathrm{Im}(T-I)^3) = 2$. Find the Jordan canonical form of T.

(a) The dimension of the generalized eigenspace K_1 equals the multiplicity of eigenvalue $\lambda = 1$, which is 4. The possible dot diagrams are:

$$\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}, \quad \begin{array}{cc} \bullet & \bullet \\ \bullet & \\ \bullet & \\ & \end{array}, \quad \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ & \\ & \end{array}, \quad \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & & \\ & & \\ & & \end{array}, \quad \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ & & & \\ & & & \\ & & & \end{array}.$$

(b) By rank-nullity,

$$\dim\ker(T-I)^3 = 5 - \dim(\mathrm{Im}(T-I)^3) = 3.$$

Thus $\dim\ker(T-I)^3 < \dim K_1$**, so that** $(T-I)^3_{K_1} \neq 0$**. Thus the in any basis of** K_1 **consists of a union of disjoint cycles of generalized eigenvectors, there much be a cycle of length at least 4. Thus the dot diagram for** K_1 **must be the first one above (with a cycle of length 4). The Jordan canonical form is**

$$\begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & \\ & & & & 0 \end{pmatrix}.$$

(Note that since the multiplicity of eigenvalue $\lambda = 0$ **is 1, we have** $\dim(K_0) = \dim(E_0) = 1$**, so there is only one** 1×1 **Jordan block corresponding to eigenvalue zero.)**

3. [10 points] Let V be the real vector space spanned by the polynomials 1, x, y, and xy. Let $D_x : V \to V$ (resp. $D_y : V \to V$) be differentiation with respect to x (resp. y). Let $T = D_x - D_y$. Find the Jordan canonical form J of T and a basis β of V such that $[T]_\beta = J$.

Since the total degree of $T(f)$ is less than that of f for every nonzero $f \in V$, the map T is nilpotent. It follows that the only eigenvalue is zero and the characteristic polynomial is $(-t)^{\dim(V)} = t^4$. Let us find the Jordan canonical form first.

$$\mathrm{Im}(T) = \mathrm{span}\{T(1), T(x), T(y), T(xy)\} = \mathrm{span}\{1, y - x\},$$

$$\mathrm{Im}(T^2) = \mathrm{span}\{T(y - x)\} = \mathrm{span}\{1\},$$

and $T^3 = 0$. Thus the longest cycle (for eigenvalue 0) in a Jordan basis has length 3. Since $\dim(V) = 4$, this determines the Jordan canonical form:

$$J = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

(with a 3×3 Jordan block and a 1×1).

As for finding a corresponding Jordan basis, we have

$$T^2(xy) = T(y - x) = -2.$$

Let us take $\{-2, y - x, xy\}$ as the cycle of length 3 in our Jordan basis. Since $\ker(T) = \mathrm{span}\{1, x + y\}$, we take $\{x + y\}$ as the cycle of length 1. Setting

$$\beta = \{-2, y - x, xy\} \cup \{x + y\} = \{-2, y - x, xy, x + y\}$$

we have $[T]_\beta = J$.

Extra space for Question 3

Question 4 is on the next page.

4. [10 points] Let $A \in M_{2\times 2}(\mathbb{C})$ be invertible. Show that for every positive integer n, there exists a matrix $B \in M_{2\times 2}(\mathbb{C})$ such that $B^n = A$.

Since $\mathbb{C}$ is algebraically closed, A has a Jordan canonical form over $\mathbb{C}$. The possibilities for the Jordan canonical form are

$$\text{(i)} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \quad (\lambda_1 \neq \lambda_2), \quad \text{(ii)} \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}, \quad \text{(iii)} \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}.$$

(Case (i) corresponds to the case when A has two distinct eigenvalues, while (ii) and (iii) corresponds to the case where A has an eigenvalue of multiplicity 2.) Combining the case (i) and (ii) above, thus there exists a matrix $P \in M_{2\times 2}(\mathbb{C})$ and a matrix $J \in M_{2\times 2}(\mathbb{C})$ of the form

$$(1) \begin{pmatrix} \lambda & \\ & \lambda' \end{pmatrix} \quad \text{or of the form} \quad (2) \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}.$$

such that $PJP^{-1} = A$. To show that A has an n-th root, it is enough to show that J has an n-th root, as if $X^n = J$, then $(PXP^{-1})^n = PX^nP^{-1} = A$. If J is of the form (1), then

$$X = \begin{pmatrix} \lambda^{1/n} & \\ & \lambda'^{1/n} \end{pmatrix}$$

(where for $z \in \mathbb{C}$, $z^{1/n}$ means an n-th root of z) satisfies $X^n = J$. Suppose J is of the form (2). Since A is invertible, zero is not an eigenvalue of A and hence $\lambda \neq 0$. Thus we can write J as

$$J = \lambda \begin{pmatrix} 1 & 1/\lambda \\ & 1 \end{pmatrix}.$$

If we take

$$X = \lambda^{1/n} \begin{pmatrix} 1 & 1/(n\lambda) \\ & 1 \end{pmatrix},$$

then in view of

$$\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ & 1 \end{pmatrix},$$

we easily see that $X^n = J$.

5. [10 points] Let F be a field and n a positive integer.

(a) Let V be an n-dimensional vector space over F and T a linear operator on V whose characteristic polynomial $p_T(t)$ is irreducible in $F[t]$. Show that there exists a basis β of V such that $[T]_\beta$ is the companion matrix for the polynomial $p_T(t)$. That is, if $p_T(t) = (-1)^n(t^n + \sum_{i=0}^{n-1} a_i t^i)$, then

$$[T]_\beta = \begin{pmatrix} 0 & & & \dots & & 0 & -a_0 \\ 1 & 0 & & \dots & & 0 & -a_1 \\ 0 & 1 & 0 & \dots & & 0 & -a_2 \\ & & & & & & \\ \vdots & \vdots & & & & \vdots & \\ & & & & 1 & 0 & -a_{n-2} \\ 0 & 0 & & & 0 & 1 & -a_{n-1} \end{pmatrix}.$$

(Hint: Let v be a nonzero element of V. Consider the T-cyclic subspace generated by v.)

(b) Let A and B be $n \times n$ matrices with entries in F whose characteristic polynomials are equal and irreducible in $F[t]$. Use Part (a) to deduce that A and B are similar (over F).

(a) Let v be a nonzero element of V. Let W be the T-cycle subspace generated by v. Then W is a nonzero T-invariant subspace of V. Since $p_T(t)$ is irreducible, then the only T-invariant subspaces of V are zero and V. Thus $W = V$. Now it follows from $\dim(W) = n$ that $\beta = \{v, T(v), \dots, T^{n-1}(v)\}$ is a basis of W (and hence V). If $T^n(v) + \sum_{i=0}^{n-1} b_i T^i(v) = 0$, then we have

$$[T]_\beta = \begin{pmatrix} 0 & & & \dots & & 0 & -b_0 \\ 1 & 0 & & \dots & & 0 & -b_1 \\ 0 & 1 & 0 & \dots & & 0 & -b_2 \\ & & & & & & \\ \vdots & \vdots & & & & \vdots & \\ & & & & 1 & 0 & -b_{n-2} \\ 0 & 0 & & & 0 & 1 & -b_{n-1} \end{pmatrix},$$

Extra space for Question 5

Question 6 is on the next page.

and $p_T(t) = (-1)^n(t^n + \sum_{i=0}^{n-1} b_i t^i)$**. Thus we must have** $a_i = b_i$ **for all** i **and the basis** $\beta = \{v, T(v), \ldots, T^{n-1}(v)\}$ **is the desired basis of** V**.**

(b) If the characteristic polynomial of A **and** B **is equal to** $(-1)^n(t^n + \sum_{i=0}^{n-1} a_i t^i)$**, applying part (a) to the maps** $L_A, L_B : F^n \to F^n$**, we see that both** A **and** B **are similar to the matrix given in the statement of part (a). Thus** A **and** B **are similar to each other.**

6. [7 points] Let T be a diagonalizable linear operator on a finite-dimensional vector space V. Let S be a linear operator on V such that every eigenspace $E_\lambda(T)$ of T is S-invariant. Show that $TS = ST$. (Hint: Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T. Use the decomposition $V = \bigoplus_{i=1}^{k} E_{\lambda_i}(T)$.)

Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T. Since T is diagonalizable, we have $V = \bigoplus_{i=1}^{k} E_{\lambda_i}(T)$. Given $v \in V$, thus there exist unique $v_1, \ldots, v_k$, with $v_i \in E_{\lambda_i}(T)$, so that $v = \sum_{i=1}^{k} v_i$. By linearity of T and S,

$$ST(v) = S(T(v)) = \sum_{i=1}^{k} S(T(v_i)).$$

Since $v_i \in E_{\lambda_i}(T)$, we have $T(v_i) = \lambda_i v_i$. Thus

$$ST(v) = \sum_{i=1}^{k} S(\lambda_i v_i)) = \sum_{i=1}^{k} \lambda_i S(v_i)).$$

Since each $E_{\lambda_i}(T)$ is S-invariant, $S(v_i) \in E_{\lambda_i}(T)$, so that $\lambda_i S(v_i) = T(S(v_i))$. Thus

$$ST(v) = \sum_{i=1}^{k} T(S(v_i))) = TS(v),$$

by linearity of T and S.

7. [bonus, 7 points] Let T be a diagonalizable linear operator on a finite-dimensional vector space V over a field F. Let S be the set of eigenvalues of T and $\lambda \in S$ a fixed eigenvalue. Let $\pi : V \to V$ be the projection map onto E_λ, relative to the decomposition $V = \bigoplus_{\mu \in S} E_\mu$. Show that there exists a polynomial $f(t) \in F[t]$ such that $\pi = f(T)$.

Let

$$g(t) = \prod_{\mu \in S - \{\lambda\}} (t - \mu).$$

Then $g(\lambda) \neq 0$. **Set**

$$f(t) = \frac{1}{g(\lambda)} g(t).$$

We claim that $f(T) = \pi$. **Indeed, in view of the decomposition** $V = \bigoplus_{\mu \in S} E_\mu$ **, it is enough to check that** $f(T)(v) = 0$ **if** $v \in E_\mu$ **for** $\mu \in S - \{\lambda\}$**, and** $f(T)(v) = v$ **if** $v \in E_\lambda$**. Note that if** $p(t) = \sum a_i t^i$ **is any polynomial and** $v \in E_\mu$ **with** μ **any scalar, then** $p(T)(v) = \sum a_i T^i(v) = \sum a_i \mu^i v = p(\mu)v$**. If** $\mu \in S - \{\lambda\}$**, then it is clear from the definition of** $f(t)$ **that** $f(\mu) = 0$**, so that** $f(T)(v) = f(\mu)v = 0$ **for any** $v \in E_\mu$**. On the other hand, since** $f(\lambda) = 1$**, for any** $v \in E_\lambda$**, we have** $f(T)(v) = f(\lambda)v = v$**, as desired.**

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