MAT 257: Analysis II.

Handout 1: week of September 12-16, 2011.

I. Assignment.

Note that this is *not* to be handed in. The homework that will be collected appears in Section III below, on page 2.

Over the next week. Read pages 1-10 of Spivak, and work the problems on pages 4-5. Some of these are contained in the assignment below that is to be handed in on September 21.

Make sure to solve *in detail* all of the starred exercises (problems 1, 10, 12, 13).

Problem 8b in Spivak is incorrect as stated. Problems 8b and 8c in Spivak should thus be replaced by: "Show that T is angle-preserving if and only if there exists some $\lambda > 0$ such that $|Tx| = \lambda|x|$ for all $x \in \mathbb{R}^n$."

II. Summary.

The *Euclidean inner product* of points x and y in $\mathbb{R}^n$ is defined by

$$\langle x, y\rangle = \sum_{i=1}^{n} x^i y^i.$$

The *Euclidean norm* of $x \in \mathbb{R}^n$ is defined by $|x| = \langle x, x\rangle^{1/2}$.
The *Euclidean distance* between points x and y in $\mathbb{R}^n$ is defined by $d(x, y) = |x - y|$.

(In this course, we will often omit the word "Euclidean" in all of the above.)

More generally, if V is any vector space over $\mathbb{R}$, then *an inner product on* V is any function $V \times V \to \mathbb{R}$ that is symmetric, bilinear, and positive definite (these are conclusions (1), (2), and (3) of Theorem 1-2 in Spivak.) And a *norm on* V is any function $V \to [0, \infty)$ that is positive definite, positively homogeneous of degree 1, and subadditive (these are conclusions (1), (4), and (3) of Theorem 1-1 in Spivak.)

The adjective *Euclidean* is needed if we want to distinguish different norms etc on different spaces or sets, or to emphasize that we are considering norms/inner products etc that are not necessarily Euclidean. We will not do this much in this class, but we will do it on this assignment.

Let[1] $(x, y) \mapsto \langle x, y\rangle_*$ be an inner product on a vector space V, and define $|x|_* : V \to [0, \infty)$ by $|x|_* = \langle x, x\rangle_*^{1/2}$. The proof of the Cauchy-Schwarz inequality (for the Euclidean norm and inner product) applies *with absolutely no change* to this situation to show that $|\langle x, y\rangle_*| \leq |x|_* \, |y|_*$, and hence, again by the same proof as before, that the function $x \mapsto |x|_*$ is indeed a norm.

A related notion is that of a *metric*, which is discussed in one of the homework problems below.

Later we will see that many aspects of the topology of $\mathbb{R}^n$ hold for more general spaces called "metric spaces" (of which $\mathbb{R}^n$ is the basic example).

An alternative proof of Cauchy-Schwarz. The hardest part in the above is the verification of the subadditivity of the Euclidean norm (as in Spivak) or more generally of the $|x|_*$ norm as above. In either case, this relies on the Cauchy-Schwarz inequality $|\langle x, y\rangle| \leq |x| \; |y|$. Here is a proof of of Cauchy-Schwarz that is slightly different from the one in the book. (It is written for the Euclidean norm and inner product but is exactly the same for a more general inner product and the associated norm.)

Step 1. By rewriting the inequality $0 \leq |x \mp y|^2 = \langle x - y, x - y\rangle$, we find that

$$\pm\langle x, y\rangle \leq \frac{1}{2}(|x|^2 + |y|^2), \qquad \text{so that } |\langle x, y\rangle| \leq \frac{1}{2}(|x|^2 + |y|^2). \tag{1}$$

[1] The notation "$(x, y) \in V \times V \mapsto \langle x, y\rangle_* \in \mathbb{R}$ " means "the function that takes the pair $(x, y) \in V \times V$ as input and produces as output a real number denoted $\langle x, y\rangle_*$".

Step 2. Note that the substitution $x \to \alpha x, y \to \frac{1}{\alpha} y$, for $\alpha > 0$, converts (1) into

$$|\langle x, y\rangle| \le \frac{1}{2}(\alpha^2 |x|^2 + \alpha^{-2}|y|^2). \tag{2}$$

Step 3. Now (2) is a 1-parameter family of inequalities. The best of these is the one where the right-hand side is as small as possible. We can use calculus to find that this happens when $\alpha^2 = |y|/|x|$. Substituting this into (2), we find that $|\langle x, y\rangle| \le |x|\ |y|$.

Step 4. By undoing these arguments, we find that if $|\langle x, y\rangle| = |x|\ |y|$ and x and y are nonzero, then $0 = |\alpha x + \alpha^{-1} y|^2$ or $0 = |\alpha x - \alpha^{-1} y|^2$ for $\alpha^2 = |y|/|x|$, so that x and y are linearly dependent. On the other hand, if either x or y equals zero, then it is clear that x and y are linearly dependent and that $|\langle x, y\rangle| = |x|\ |y|$. □

If you think about it, this proof is in some sense the same as the standard one (cf Spivak), but it is conceptually a little different. Here, the main point is that if we have an inequality like (1) where "one side has more symmetry than the other" (ie, we can manipulate the inequality in a way that leaves one side unchanged but introduces a free parameter on the other side), we can exploit that to improve the inequality. In this instance the improved inequality we obtain from (1) is Cauchy-Schwarz.

III. Hand in on September 21.

(1) As noted above, every inner product yields a norm. The converse is *not* true, as you will show in this exercise.
   (a) Let $\langle x, y\rangle_*$ denote an inner product on a vector space V, and let $|x|_* := \langle x, x\rangle_*^{1/2}$. Prove that if $|x|_* = |y|_* = 1$ and $x \ne y$, then $|\theta x + (1-\theta) y|_* < 1$ for any $\theta \in (0, 1)$. Give a geometric interpretation (of at most two sentences, accompanied by a picture if you like).
   (b) For $x \in \mathbb{R}^n$, define $|x|_{sum} := \sum_{i=1}^n |x^i|$. Verify that this function is a norm on $\mathbb{R}^n$.
   (c) Prove that there does not exist any inner product $\langle x, y\rangle_{sum}$ with the property that $|x|_{sum} = \langle x, x\rangle_{sum}^{1/2}$ for all $x \in \mathbb{R}^n$.

(2) Let V be a vector space, and let $|x|_*$ denote a norm on V. Define $d_*(x, y) = |x - y|_*$.
   (a) Prove that d_* satisfies:

$$d_*(x, y) \ge 0, \text{ with equality iff } x = y,$$
$$d_*(x, y) = d_*(y, x),$$
$$d_*(x, z) \le d_*(x, y) + d_*(y, z)$$

   for all x, y, z in V .
   (b) If S is any set, then a function $d_* : S \times S \to [0, \infty)$ is said to be a *metric on S* if it satisfies the three properties given above. So in part (a) you have shown that every norm yields a metric. Prove that the converse is false, i.e. that there exists a set S and a metric on S that does not arise from any norm via the construction of part (a).

(3) Exercise 1-6 on page 4 of Spivak

(4) Exercise 1-7 on page 4 of Spivak.