MAT 257, Handout 10: November 14-18, 2011.

Assignment. Read the parts of Spivak that discuss integrability and Fubini's Theorem, and work the exercises.

Some additional exercises (not to be handed in).

1. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function and that p is a regular value of f. Let $M := f^{-1}(p)$. Prove that M has measure 0.

2. Deduce that a hyperplane (i.e., a set of the form $\{x : \langle x, p\rangle + b = 0\}$ for some nonzero $p \in \mathbb{R}^n$ and some $b \in \mathbb{R}$) has measure 0.

This is an immediate corollary of exercise **1**, but it is worth pointing out because of its importance. Also, in thinking about exercise **1**, it may help to consider this first, since one of the issues you must address in exercise **1** is the possible non-compactness of M, and a hyperplane is a particularly simple example of a noncompact manifold.

3. Prove that the set $\{x = (x^1, \dots, x^n) \in \mathbb{R}^n : x^i \text{ is rational for some } i\}$ has measure 0.

4. **a**. Returning to the situation of problem **1**, give an example of a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ and a *critical* value p of f, such that $M := f^{-1}(p)$ has measure 0.

b. And now give an example of a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ and a *critical* value p of f, such that $M := f^{-1}(p)$ *does not* have measure 0.

5. Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is *any* function, then $\{p \in \mathbb{R} : f^{-1}(p) \text{ does not have measure } 0\}$ is finite or countable.