MAT 257, Handout 10: November 21-25, 2011.

Assignment. We will finish what is left of chapter 3 (partitions of unity and changes of variables) by the end of the term. So you should read that material over the next two weeks.

Solution to some exercises from homework 4. Part of Homework 4 deals with the following

Theorem 1. *Assume that $g : \mathbb{R}^n \to \mathbb{R}^k$ is a continuously differentiable function, and that $p \in \mathbb{R}^k$ is such that the Jacobian matrix $g'(x)$ has rank k at every point in $g^{-1}(p)$, so that $M := g^{-1}(p)$ is a C^1 manifold of dimension $n-k$.*

Assume also that $F : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Let f denote the restriction of F to M. If f has a local maximum or minimum at a point $x_0 \in M$, then there exist $\lambda_1, \ldots, \lambda_k$ such that

$$D_j F(x_0) = \sum_{i=1}^{n} \lambda_i D_j g^i(x_0) \qquad \text{for } j = 1, \ldots, n.$$

The proof contains exercises 2 and 3 from homework 4.

Proof. **1**. We first consider the special case in which there exists an open set $U \subset \mathbb{R}^n$ containing x_0 such that $U \cap M = \{x \in U : x^{n-k+1} = \cdots x^n = 0\} = U \cap (\mathbb{R}^{n-k} \times \{0\})$.

Then there exists some $t_0 > 0$ such that $x_0 + te_i \in M$ for all t such that $|t| < t_0$ and all $i = 1, \ldots, n-k$. Thus the function $\phi_i(t) = f(x_0 + te_i)$ is well-defined for $i \in \{1, \ldots, n-k\}$ and $|t| < t_0$, and our hypotheses imply that ϕ_i has a local minimum or maximum at $t = 0$. Thus $D_i f(x_0) = \phi_i'(0) = 0$. So the Jacobian "matrix" $f'(x_0)$ (here a row vector) has the form

$$f'(x_0) = (0, \cdots, 0, D_{n-k+1} f(x_0), \cdots, D_n f).$$

Next, recall that $g(y) = p = (p^1, \ldots, p^k)$ for every $y \in M$ (by definition of M). So if we define $\psi_j^i(t) = g^i(x_0 + te_j)$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n-k$, then $\psi_j^i(t) = p^i$ whenever $|t| < t_0$, and it follows that

$$0 = (\psi_j^i)'(0) = D_j g^i(t) \qquad \text{for } i = 1, \ldots, k \text{ and } j = 1, \ldots, n-k.$$

That is, the Jacobian matrix $g'(x_0)$ has the form

$$g'(x_0) = \begin{pmatrix} 0 & \cdots & 0 & D_{n-k+1} g^1(x_0) & \cdots & D_n g^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & D_{n-k+1} g^k(x_0) & \cdots & D_n g^k \end{pmatrix}$$

So it is clear that the row-space of g' (the space of linear combinations of its rows) is contained in $V := \{\text{row vectors } v = (v^1, \ldots, v^n) : v^1, \ldots, v^{n-k} = 0\}$, which is a k-dimensional vector space. But by assumption, the rank of $g'(x_0)$ is k, so the row-space of $g'(x_0)$ is k-dimensional, and thus it must be the case that the row-space of g' equals V. Since we have shown that $f'(x_0) \in V$, it follows that $f'(x_0)$ is a linear combination of the row vectors $(g^1)'(x_0), \ldots, (g^k)'(x_0)$, which is what we have to show.

2. Now consider the general case. An approach that is conceptually simple is just to reduce this to the previous case using a diffeomorphism to "flatten out" the manifold M near the point x_0. Here are some details:

Consider a point x_0 where F has a local maximum or minimum, and let h be a C^1 diffeomorphism between an open set $U \subset \mathbb{R}^n$ containing x_0 and another open set V in $\mathbb{R}^n$, such that

$$h(M \cap U) = V \cap (\mathbb{R}^{n-k} \times \{0\}).$$

Let $\tilde{f} = f \circ h^{-1}$, and let $\tilde{g} = g \circ h^{-1}$.

We claim that $\tilde{f}$ and $\tilde{g}$ satisfy the hypotheses of the theorem in the special case already considered above at the point $z_0 := h(x_0)$. (This is a routine verification. Briefly: differentiability is clear, the fact that z_0 is a local maximum/minimum of $\tilde{f}$ is a consequence of h being a diffeomorphism between $M \cap U$ and $V \cap (\mathbb{R}^{n-k} \times \{0\})$, and the fact that $\tilde{g}'(z_0) = g'(x_0)(h^{-1})'(z_0)$ has rank k follows from the corresponding property for $g'(x_0)$ together with the fact that, since h is a diffeomorphism, $(h^{-1})'(z_0)$ is nonsingular.)

Thus it follows from the special case that there exist $\lambda_1, \ldots, \lambda_k$ such that $\tilde{f}'(z_0) = \sum_{j=1}^k \lambda_j (\tilde{g}^j)'(z_0)$.

By the chain rule, and since $h^{-1}(z_0) = x_0$,

$$\tilde{f}'(z_0) = f'(h^{-1}(z_0))(h^{-1})'(z_0) = f'(x_0)(h^{-1})'(z_0)$$

Similarly $(\tilde{g}^i)'(z_0) = (g^i)'(x_0)(h^{-1})'(z_0)$. Hence we deduce that

$$f'(x_0)(h^{-1})'(z_0) = \sum_{j=1}^k \lambda_j (g^j)'(x_0)(h^{-1})'(z_0).$$

Finally, note that since h is a diffeomorphism, $(h^{-1})'(z_0)$ is an invertible matrix (with inverse $h'(x_0)$ in fact) so we can multiply both sides of the above identity on the right by $h'(x_0)$ to find that

$$f'(x_0) = \sum_{j=1}^k \lambda_j (g^j)'(x_0).$$

which is what we have to show. □

Homework 5, due Wednesday December 7.

There are two options, **Option A** and **Option B**. Please write prominently on the first page of your assignment **Option A** or **Option B**, depending on which you choose.

Option A is:

1. Spivak, exercise 3-9. To receive full marks, you must use at most 2 sentences for part (a), and at most one sentence for part (b). [4 marks]

2. Spivak, exercise 3-13. For part (a), try to deduce the conclusion you want from a more general statement (that you must formulate and prove.) Toward this end, note that the collection of rectangles being considered here can be identified with a subset of the product space $\mathbb{Q}^{2n}$. [8 marks, 4 for each half]

3. Spivak, exercise 3-22. [7 marks]

4. Spivak, exercise 3-27. [7 marks]

5. Spivak, exercise 3-32. (It is my painful duty to point out that Spivak has misspelled the name "Leibniz".) [7 marks]

6. Spivak, exercise 3-33. You should need 1- 2 sentences for each of parts (a) and (b). [4 marks]

7. Spivak, exercise 3-34. Up to 3 sentences are needed for this. [3 marks]

Option B is:

problems 1, 2, 3, 4, 5 above (with the same number of marks as above), together with

$\star$. Without handing them in, solve problems 3-12 and 3-20 in Spivak. Then write up and hand in a solution to the following problem:

Let $A = [a, b] \times [c, d]$. We say that a function $f : A \to \mathbb{R}$ is *nondecreasing in the x variable* if for every $y \in [c, d]$, the function $g_y : [a, b] \to \mathbb{R}$ defined by $g_y(x) = f(x, y)$ is nondecreasing.

If f is bounded and nondecreasing in the x variable, is it necessarily integrable? Give a proof or a counterexample.

[7 marks]