MAT 257, Handout 13: December 5 - 7, 2011.
The Change of Variables Theorem.

In these notes, I try to make more explicit some parts of Spivak's proof of the Change of Variable Theorem, and to supply most of the missing details of points that I think he glosses over too quickly. Our goal is:

Theorem 1. *Suppose that A is an open subset of $\mathbb{R}^n$ and that $g : A \to \mathbb{R}^n$ is a diffeomorphism onto its image.*

Then for every integrable $f : g(A) \to \mathbb{R}$, the function $(f \circ g)|\det g'|$ is an integrable function $A \to \mathbb{R}$, and

$$(1) \qquad \int_A f \circ g|\det g'| \;=\; \int_{g(A)} f.$$

The following terminology will be convenient.

Definition 1. *Given a diffeomorphism $g : A \to g(A)$ as above, and a subset $V \subset A$, we will say that "*g has the change of variables property on V*" if the conclusions of the theorem hold, with A replaced by V, for every integrable $f : g(V) \to \mathbb{R}$.*

1. PROPERTIES OF DIFFEOMORPHISMS

We need several general properties of diffeomorphisms. The first is a basic fact that is used very often.

Theorem 2. *Suppose that A is an open subset of $\mathbb{R}^n$ and that $g : A \to \mathbb{R}^n$ is a diffeomorphism onto its image. If Z is a subset of A with measure 0, then $g(Z)$ also has measure 0.*

Proof. **1.** First, for every point $x \in A$, let R_x be an open rectangle containing x, and such that the closure[1] of $\bar{R}_x$ is contained in A. These clearly exist, since A is open.

Recall that A can be written as a countable union of compact sets. (In fact $A = \cup_{k=1}^{\infty}\{x \in A : |x| \le k, \quad \mathrm{dist}(x, \partial A) \ge 1/k\}$.) Each of these compact sets is covered by finitely many of the open rectangles R_x, by compactness, and so it follows that we can write A as a countable union of closed rectangles $A = \cup_{i=1}^{\infty}\bar{R}_{x_i}$.

[1]Recall that the closure of a set $S \subset \mathbb{R}^n$ is the union of its boundary and its interior, or equivalently, the complement of its exterior. Since the exterior is always open, it is clear that the closure is closed, as the name suggests.

Now for every i, let $Z_i := Z \cap \bar{R}_{x_i}$. Then $g(Z) \subset \cup_{i=1}^{\infty} g(Z_i)$. Since a countable union of sets of measure 0 again has measure 0, it suffices to show that $g(Z_i)$ has measure 0 for every i, and for this, it suffices to show that if S is a closed rectangle contained in A, then $g(Z \cap S)$ has measure 0.

2. Let S be a closed rectangle contained in A. We claim that there exists some number M_1 (possibly depending on S) such that

$$|g(x) - g(y)| \leq M_1 |x - y| \tag{2}$$

for all $x, y \in S$. To see this, note that since g is continuously differentiable, every partial derivative $D_i g^j$ is a continuous function, and hence[2] there exists some number M_0 such that $|D_i g^j(x)| \leq M_0$ for every $x \in S$. Then the claim (2) follows from Lemma 2-10 in Spivak, with $M_1 = n^2 M_0$.

3. We next claim that there exists some number M_2, possibly depending on S, such that if Q is any cube, then

$$\text{there exists a cube } Q' \text{ such that } g(Q \cap S) \subset Q' \text{ and } \ v(Q') \leq M_2\, v(Q). \tag{3}$$

Indeed, let ℓ denote the side-length of Q. Then it is easy to check that $|x - y| \leq \sqrt{n}\ell$ for any $x, y \in Q$, so (2) implies that $|g(x) - g(y)| \leq \sqrt{n} M_1 \ell$ for any $x, y \in Q \cap S$. So if we fix some $x \in Q \cap S$, then

$$\begin{aligned} g(Q \cap S) &\subset \{z : |z - g(x)| \leq \sqrt{n} M_1 \ell\} \\ &\subset \text{ cube with centre } g(x) \text{ and side-length } 2\sqrt{n} M_1 \ell =: Q'. \end{aligned}$$

Then (3) follows, with $M_2 = (2\sqrt{n} M_1)^n$.

4. Let $Z' = Z \cap S$, and fix $\varepsilon > 0$. We next claim that there exists a countable family of closed cubes Q_j such that

$$Z' \subset \cup_{j=1}^{\infty} Q_j, \qquad \sum_{j=1}^{\infty} v(Q_j) < \varepsilon. \tag{4}$$

The point is that we can use cubes rather than rectangles in the definition of a set of measure 0. This is true since, given any closed rectangle R, we can clearly find a finite collection of closed cubes $Q_1, \ldots Q_k$ such that $R \subset \cup_{i=1}^{k} Q_i$ and $\sum_{i=1}^{k} v(Q_i) \leq 2v(R)$. (If this is not clear, consider it as an exercise.) So we may first cover Z' by rectangles whose volumes sum to less than $\varepsilon/2$, and then cover each such rectangle by a finite number of cubes whose volumes add up to at most twice the volume of the rectangle. The collection of cubes obtained in this way satisfies (4).

[2] The image of a compact set via a continuous function is compact, and hence bounded.

5. Let $\{Q_j\}$ be a family of cubes satisfying (4), and for each j let Q_j' be a cube related to Q_j as in (3). Then it follows immediately that

$$g(Z') = g(Z \cap S) \subset g(\cup_{j=1}^{\infty}(Q_j \cap S)) = \cup_{j=1}^{\infty} g(Q_j \cap S) \subset \cup_{j=1}^{\infty} Q_j'$$

and

$$\sum_{j=1}^{\infty} v(Q_j') \leq \sum_{j=1}^{\infty} M_2 v(Q_j) < M_2 \varepsilon.$$

Since $M_2\varepsilon$ can be made arbitrarily small, this proves that $g(Z')$ has measure 0, and hence by Step 1 completes the proof of the theorem. □

The above theorem is the main point in the proof of the following:

Lemma 1. *Suppose that A is an open subset of $\mathbb{R}^n$ and that $g : A \to \mathbb{R}^n$ is a diffeomorphism onto its image.*

If $f : g(A) \to \mathbb{R}$ is locally integrable (ie, on every compact subset K of $g(A)$, f is bounded, with a discontinuity set of measure 0) then $f \circ g|\det g'|$ is locally integrable on g.

In particular, $f \circ g|\det g'|$ is Riemann integrable (that is, the supremum of the lower sums equals the infimum of the upper sums) on any compact, Jordan-measurable subset of A.

Proof. Assume that $f : g(A) \to \mathbb{R}$ is locally integrable, and fix any compact $K \subset A$.

1. The continuous image of a compact set is compact, so $g(K)$ is a compact subset of $g(A)$. Thus there exists some M such that $f(x)| \leq M$ for all $x \in g(K)$, and it follows that $|f \circ g(y)| = |f(g(y)| \leq M$ for all $y \in K$. Since $|\det g'|$ is continuous (and hence bounded on compact subsets) it follows that $(f \circ g)|\det g'|$ is bounded on K.

2. Next, note that if f is continuous at some point $g(y)$ for some $y \in K$, then $(f \circ g)|\det g'|$ is continuous at y (since g and g' are continuous everywhere). It follows that

$$\{y \in K : (f \circ g)|\det g'| \text{ is discontinuous at } y\} \subset g^{-1}(Z)$$

for

$$Z := \{x \in g(K) : f \text{ is discontinuous at } x\}.$$

But Z has measure 0 by hypothesis, and g^{-1} is a differomorphism, so it follows from Theorem 2 that $g^{-1}(Z)$ has measure 0. this proves that $(f \circ g)|\det g'|$ is integrable on K. Since K was an arbitrary compact subset of A, it follows that $(f \circ g)|\det g'|$ is locally integrable on A.

3. The final conclusion now follows from 3-8 in Spivak, (the characterization of integrable functions on a rectangle) together with the definition of a Jordan-measurable set. □

We will again use Theorem 2 in the proof of the following lemma.

Lemma 2. *Suppose that A is an open subset of $\mathbb{R}^n$ and that $g : A \to \mathbb{R}^n$ is a diffeomorphism onto its image.*

If K is a compact, Jordan-measurable subset of A, then $g(K)$ is also compact and Jordan-measurable.

Proof. If K is compact, then (since a diffeomorphism is certainly continuous, and the continuous image of a compact set is always compact) it follows that $g(K)$ is compact.

Also, we claim that the fact that g is a diffeomorphism implies that

$$\text{boundary } g(K) = g(\text{boundary } K). \tag{5}$$

Indeed, suppose that $x \in$ boundary $g(K)$, and let $y = g^{-1}(x)$. For any open ball B containing y, the set $g(B)$ is an open set (since g^{-1} is continuous) that contains x. Thus, since $x \in \text{boundary}\, g(K)$, $g(B)$ must contain both points of $g(K)$ and points of $g(K)^c$. But since g is one-to-one,

$$z \in K \iff g(z) \in g(K).$$

and so it follows that B contains both points of K and points of K^c. Since B was an arbitrary ball containing y, we conclude that $y \in \text{boundary}\, K$, and hence that boundary $g(K) \subset g(\text{boundary } K)$. The opposite inclusion follows by exactly the same argument.

Since K is Jordan measurable, $\text{boundary}\, K$ has measure 0, and hence it follows from (5) and Theorem 2 that $\text{boundary}\, g(K)$ has measure 0. Thus $g(K)$ is Jordan measurable.

□

2. REDUCTIONS AND PRELIMINARY RESULTS

Lemma 3. *If $g : \mathbb{R}^n \to \mathbb{R}^n$ is a linear function, then*

$$\int_S |\det g'| = \int_{g(S)} 1 \qquad \text{for every closed rectangle } S \subset U. \tag{6}$$

This lemma may be seen as the underlying reason that determinants appear in the change of variables formula.

Proof. Fix a linear function $g : \mathbb{R}^n \to \mathbb{R}^n$ and let N be the matrix representing g, so that $g(x) = Nx$ for every $x \in \mathbb{R}^n$ (where elements of $\mathbb{R}^n$ are viewed as column vectors.) Then the lemma asserts that

$$v(g(S)) = |\det N| v(S) \tag{7}$$

for every rectangle S.

This is an exercise in Spivak. I also sketched an alternate proof in the lecture, which is given below. (Note, I have not written out the details of the proof sketched in Spivak's exercise, but it is probably easier than the proof given here, especially since i have omitted some details from a key induction argument.)

Note that if g does not have full rank (i.e. if $\det N = 0$), then both sides of (7) equal 0. Thus (7) clearly holds in this case, so we assume for the duration of the proof that $\det N \neq 0$.

Claim 1. There exist $n \times n$ matrices Q and T such that Q is orthogonal (that is, $QQ^T = Q^TQ = I$, where I denotes the identity matrix) and T is upper triangular (that is, $T_{ij} = 0$ if $i > j$), and $N = QT$.

In fact, as I showed, we can construct such a matrix Q as follows: Let $v_1, \ldots, v_n$ denote the columns of N. Inductively choose *orthonormal* (column) vectors $q_1, \ldots, q_n$ such that

$$\{v_1, \ldots, v_i\} \subset \operatorname{span}(q_1, \ldots, q_i), \tag{8}$$

and let Q be the matrix whose columns are $q_1, \ldots, q_n$ (in that order).

Then the fact that the vectors $\{q_i\}_{i=1}^n$ are orthonormal implies that $Q^TQ = QQ^T = I$, and (8) implies that $Q^TN := T$ is upper triangular. Then the identity $N = QT$ follows.

Claim 2. Then it follows from the fact properties of determinants that

$$1 = \det I = \det(QQ^T) = \det Q \ \det Q^T = (\det Q)^2$$

so that $|\det Q| = |\det Q^T|1$, and then that

$$|\det T| = |\det Q^TN| = |\det Q^T| \, |\det N| \ = \ |\det N|.$$

Claim 3. Now consider a rectangle

$$S = [a_1, a_1 + \ell_1] \times \cdots \times [a_n, a_n + \ell_n] = \{a + \sum_{i=1}^n \theta_i \ell_i e_i : 0 \leq \theta_i \leq 1\}$$

where $a := (a_1, \ldots, a_n)^T$ and $e_1, \ldots, e_n$ are the standard basis vectors (and we identify points in $\mathbb{R}^n$ with column vectors). Then since $Ne_i = v_i =$ the ith column of N,

$$g(S) = \{g(a) + \sum_{i=1}^n \theta_i \ell_i v_i : 0 \leq \theta_i \leq 1\}.$$

We compute the volume of this using the fact that

k-dim volume of a k-dim parallelpiped $=$ ($k-1$-dim volume of the base)$\times$ height for every k. The point is that the equation $N = QT$ states that (recalling that $\{v_i\}$ are the columns of N and that T is upper triangular)

$$v_k = \sum_{j=1}^{k} q_j T_{jk} \qquad \text{for every } k, \text{ where } \{T_{jk}\} \text{ are the entries of } T,$$

so that the height of the vector $\ell_k v_k$ above the $k-1$-plane spanned by $v_1, \ldots v_{k-1}$ (or equivalently, spanned by $q_1, \ldots, q_{k-1}$ is $\ell_k T_{kk}$. This is also the height of the k-dim parallelepiped

$$\{g(a) + \sum_{i=1}^{k} \theta_i \ell_i v_i : 0 \le \theta_i \le 1\}.$$

over its base

$$\{g(a) + \sum_{i=1}^{k-1} \theta_i \ell_i v_i : 0 \le \theta_i \le 1\}.$$

Then a straightforward induction argument leads to the conclusion that

$$v(g(S)) \ = \ \prod_{i=1}^{n} |\ell_i T_{ii}| \ = \ v(R)\ |\det T| = v(R) |\det N|.$$

which is what we had to show ☐

Lemma 4. *Suppose that A is an open subset of $\mathbb{R}^n$ and that $g : A \to \mathbb{R}^n$ is a diffeomorphism onto its image.*

Assume that A has an open cover $\mathcal{O}$ such that g has the change of variables property on U for every $U \in \mathcal{O}$.

Then g has the change of variables property on A.

Proof. This is reduction (1) in Spivak, pages 67-68 ☐

Lemma 5. *Suppose that A is an open subset of $\mathbb{R}^n$ and that $g : A \to \mathbb{R}^n$ is a diffeomorphism onto its image.*

Assume that U is an open, Jordan-measurable subset of A, and that

(9) $$\int_K |\det g'| \ = \ \int_{g(K)} 1 \qquad \text{for every compact Jordan-measurable } K \subset U.$$

Then g has the change of variables property on U.

Proof. This is essentially reduction (2) in Spivak, pages 68. More precisely, Spivak's argument shows that g has the change of variables property in U if

(10)
$$\int_K |\det g'| \;=\; \int_{g(K)} 1 \qquad \text{whenever } K = g^{-1}(R) \text{ for some closed rectangle } R \subset g(U).$$

Since every rectangle is certainly Jordan-measurable, it follows from Lemma 2 that every set of the form $K = g^{-1}(R)$, R a closed rectangle, is also Jordan-measurable, so our hypothesis (9) implies the condition (10) needed for Spivak's proof. □

Lemma 6. *Suppose that A is an open subset of $\mathbb{R}^n$ and that $g : A \to \mathbb{R}^n$ is a diffeomorphism onto its image.*

Assume that U is an open subset of A, and that

$$\int_S |\det g'| \;=\; \int_{g(S)} 1 \qquad \text{for every closed rectangle } S \subset U. \tag{11}$$

Then (9) *holds, and hence g has the change of variables property on U.*

This seems to me to be missing from Spivak's discussion. He suggests (in his discussion of reduction (4) on page 69) that it follows from reductions (1) and (2) on pages 67-68 (i.e., essentially lemmas 4 and 5 above) but this seems to me to be a bit of a stretch.

Proof. Fix a compact, Jordan-measurable $K \subset U$.

Fix a rectangle R containing K, and note that since K is Jordan measurable and $\det g'$ is continuous,

$$\begin{aligned}\int_K |\det g'| \;=\; \int_R \chi_K|\det g'| &= \sup_{\text{partitions } P} L(\chi_K|\det g'|, P) \\ &= \inf_{\text{partitions } P} U(\chi_K|\det g'|, P).\end{aligned} \tag{12}$$

where $\chi_K(x) = 1$ if $x \in K$ and 0 otherwise.

Next, for any partition P, if S is any subrectangle then it follows from the definitions that

$$m_S(\chi_K|\det g'|) = \begin{cases} m_S(|\det g'|) & \text{if } S \subset K \\ 0 & \text{if not} \end{cases},$$

and

$$M_S(\chi_K|\det g'|) = \begin{cases} M_S(|\det g'|) & \text{if } S \cap K \neq \emptyset \\ 0 & \text{if not} \end{cases}$$

Also, because K is compact and U is open, there exist partitions of R such that every subrectangle S that intersects K is contained in U. Thus, for every subrectangle S of such a partition, using hypothesis (11),

$$m_S(\chi_K|\det g'|)v(S) \le \begin{cases} \int_S |\det g'| = \int_{g(S)} 1, & \text{if } S \subset K \\ 0 & \text{if not.} \end{cases}$$

Since the sets $\{g(S) : S \subset K\}$ have disjoint interiors and are contained in $g(K)$, we conclude by summing that

$$L(\chi_K|\det g'|, P) \le \int_{g(K)} 1.$$

for every partition. Similarly

$$U(\chi_K|\det g'|, P) \ \le \ \int_{g(K)} 1$$

for every partition P. Then (11) follows from combining the last two inequalities with (12). □

Note that it follows immediately from Lemma 6 and Lemma 6 that

Lemma 7. *Every invertible linear function $\mathbb{R}^n \to \mathbb{R}^n$ has the change of variables property on every open set.*

Our final reduction is easier:

Lemma 8. *Assume that $g : A \to \mathbb{R}^n$ and $h : g(A) \to \mathbb{R}^n$ are diffeomorphisms, and that g and h both have the change of variables property on their respective domains. Then $h \circ g$ has the change of variables property on A.*

Proof. Consider any integrable $f : h \circ g(A) \to \mathbb{R}$. Since h has the change of variables property, $(f \circ h)|\det h'|$ is integrable and

$$\int_{h(g(A))} f = \int_{g(A)} (f \circ h)|\det h'|.$$

Then, since g has the change of variables property,

$$\int_{g(A)} (f \circ h)|\det h'| \ = \ \int_A ((f \circ h)|\det h'|) \circ g \ |\det g'|$$

and in particular $((f \circ h)|\det h'|) \circ g \ |\det g'|$ is integrable And

$$((f \circ h)|\det h'|) \circ g \ |\det g'| \ = \ (f \circ h \circ g)|\det h'(g)| \ |\det g'| \ = \ = \ (f \circ h \circ g)|\det(h \circ g)'|$$

by the chain rule and the fact that if M, N are matrices then $\det MN = \det M \ \det N$. Since f was an arbitrary integrable function on $h \circ g(A)$, it follows that $h \circ g$ has the change of variables property. □

3. PROOF OF THEOREM

Now we can complete the proof of the theorem.

Proof. We will prove the theorem by induction on n.

the case $n = 1$.

Suppose that A is an open subset of $\mathbb{R}$, and $g : A \to \mathbb{R}$ is continuously differentiable and one-to-one, with $g' \neq 0$ in A.

If $U \subset A$ is an open interval, then we know from the fundamental theorem of calculus that for any interval $[a, b] \subset U$,

$$\int_{[a,b]} |g'| = \int_{g([a,b])} 1, \text{ since both sides equal } \Big(\max\{g(a), g(b)\} - \min\{g(a), g(b)\}\Big).$$

Then it follows from Lemma 6 that g has the change of variables property on U, and hence from Lemma 4 that g has the change of variables property on A. This proves the theorem for $n = 1$. (Note that we could not just argue as in Spivak, pages 66-67, since the argument given there applies only to a continuous function on a single interval, whereas the theorem considers functions that are merely integrable on a set A that may be a countable union of open intervals.)

the induction step.

Now assume that the theorem holds for all integers $j \in \{1, \ldots, n-1\}$.

Claim 1: Assume that U is an open subset of $\mathbb{R}^n$ and that $h : U \to \mathbb{R}^n$ is a diffeomorphism between U and $h(U)$. Assume further that h has the form

$$h(x) = (h^1(x), \ldots, h^{n-1}(x), x^n).$$

Then h has the change of variables property in U.

proof of Claim 1. By Lemmas 6 and 4, it suffices to show that

$$\int_S |\det h'| \;=\; \int_{h(S)} 1 \quad \text{for every closed rectangle } S \subset U. \tag{13}$$

Fix such a closed rectangle S. We will write $S = S' \times [a^n, b^n]$, where S' is a rectangle in $\mathbb{R}^{n-1}$, and we will write points in S in the form $x = (y, z)$ with $y \in S'$ and $z \in [a^n, b^n]$. Then for each $z \in [a^n, b^n]$, we define the function $h_z : S' \to \mathbb{R}^{n-1}$ by

$$h_z(y) = (h^1(y, z), \ldots, h^{n-1}(y, z)).$$

Thus $h(y, z) = (h_z(y), z)$.

Note that h' has the form

$$h'(y,z) = \begin{pmatrix} D_1 h^1 & \cdots & D_{n-1}h^1 & D_n h^1 \\ \vdots & \vdots & \vdots & \vdots \\ D_1 h^{n-1} & \cdots & D_{n-1}h^{n-1} & D_n h^{n-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix} (y,z)$$

whereas $h'_z(y)$ has the form equals

$$h'_z(y) = \begin{pmatrix} D_1 h^1 & \cdots & D_{n-1}h^1 \\ \vdots & \vdots & \vdots \\ D_1 h^{n-1} & \cdots & D_{n-1}h^{n-1} \end{pmatrix} (y,z)$$

From these, if we know anything about computing determinants, we easily deduce that

$$\det h'(y,z) = \det h'_z(y). \tag{14}$$

So we can use Fubini's Theorem and the induction hypothesis to compute

$$\begin{aligned} \int_S |\det h'| &= \int_{a^n}^{b^n} (\int_{S'} |\det h'(y,z)| dy)\ dz \\ &= \int_{a^n}^{b^n} (\int_{S'} |\det h'_z(y)| dy)\ dz \\ &= \int_{a^n}^{b^n} (\int_{h_z(S')} 1\ dy)\ dz \end{aligned} \tag{15}$$

Note however that, due to the form of h,

$$\{(y,z) : z \in [a^n, b^n], y \in h_z(S')\} = h(S).$$

So Fubini's Theorem implies that

$$\int_{a^n}^{b^n} (\int_{h_z(S')} 1\ dy)\ dz = \int_{h(S)} 1.$$

This completes the proof of Claim 1. □

Claim 2 Assume that U is an open subset of $\mathbb{R}^n$ and that $k : U \to \mathbb{R}^n$ is a diffeomorphism onto its image. If k has the form

$$k(x) = k(x^1, \ldots, x^{n-1}, k^n(x))$$

then k has the change of variables property in U.

proof of Claim 2. This is very similar to the proof of Claim 1 and hence is omitted. □

To conclude the proof of the theorem, we argue that for any $a \in A$, there is an open neighborhood U of a such that $U \subset A$ and such that

(16) $g = \lambda \circ k \circ h$ in U, with λ is linear and h, k are as in claims 1 and 2 above.

By Claims 1 and 2, together with Lemmas 8 (change of variables property preserved under composition), 7 (linear functions have the change of variables property), and 4 (suffices to establish the change of variables property locally), this will complete the proof of the theorem.

To prove (16), let $\lambda = Dg(a)$, so that

$$g = \lambda \circ \tilde{g}, \qquad \tilde{g} := Dg(a)^{-1} \circ g.$$

The chain rule implies that $\tilde{g}'(a) = I$ (the identity matrix.) Next we define

$$h(x) \;=\; (\tilde{g}^1(x), \ldots, \tilde{g}^{n-1}(x), x^n).$$

Then the fact that $\tilde{g}'(a) = 1$ implies that $h'(a) = I$. Thus, by the inverse function theorem, there exists some open neighborhood $U \subset A$ such that $x \in U$ and such that h is a diffeomorphism of U onto its image $h(U)$. So we can define $k : h(U) \to \mathbb{R}^n$ by

$$k(x) = (x^1, \ldots, x^{n-1}, \tilde{g}^n(h^{-1}(x))).$$

Then the definitions imply that $\tilde{g} = k \circ h$, or equivalently that $k = \tilde{g} \circ h^{-1}$. since $\tilde{g}$ and h^{-1} are both diffeomorphisms, it folllows that k is as well. Thus we have proved (16) and completed the proof of the theorem.

□