MAT 257, Handout 14: January 9-13, 2012.

Homework 6, due January 16, 2012.

In exercise 2 below (and for the duration of this course, unless we specify otherwise), “integrable” means “integrable in the extended sense”. Recall that we have two different, equivalent characterizations of extended integrability, one from Spivak and one from Handout 12. In the exercise it is of course a good idea to use the definition of integrability, and the definition of the extended integral, that makes the exercise as easy to solve as possible.

1. Assume that A is a bounded, open subset of $\mathbb{R}^{n-1}$, and that $p \in \mathbb{R}^n$ is a point such that $p^n > 0$. Define

$$C := \{(1-s)y + sp : y \in A \times \{0\}, 0 < s < 1\}.$$

where of course $A \times \{0\}$ is the set of all points in $\mathbb{R}^n$ of the form $y = (a, 0)$ for some $a \in A$. This set is a cone with base A and vertex p.

a. [*5 marks*] Find a formula for the volume $v(C)$, involving p and $v(A)$.

b. [*5 marks*] The *centroid* of an open set $O \subset \mathbb{R}^\ell$ is defined to be the point $X \in \mathbb{R}^\ell$ whose components are

$$X^i = \frac{1}{v(O)} \int_O \pi^i, \qquad i = 1, \ldots, \ell$$

(where π^i is as usual the ith coordinate function, $\pi^i(x) = x^i$.) Find a formula for the centroid of C in terms of the centroid of A and the point p.

2. Assume that A is an open subset of $\mathbb{R}^n$, and that f and g are integrable functions $A \to \mathbb{R}$.

a. [*5 marks*] Prove that if either f or g is bounded, then the product fg is integrable.

b. [*5 marks*] Give an example to show that in general, fg need not be integrable.

3. [*10 marks*] Use Sard's Theorem (Spivak, Theorem 3-14) and the Change of Variables Theorem (Spivak, Theorem 3-13) to prove that the change of variables theorem remains true if we only assume that A is an open subset of $\mathbb{R}^n$ and that $g : A \to \mathbb{R}^n$ is continuously differentiable and injective (but does not necessarily satisfy $g'(x) \neq 0$ in A.)

Challenge problem — this can be handed in any time up to the end of the February reading week: Assume that $f : (0,1)^2 \to \mathbb{R}$ is nondecreasing in the sense that

$$f(x, y_1) \leq f(x, y_2) \text{ whenever } y_1 < y_2, \quad \text{and } f(x_1, y) \leq f(x_2, y) \text{ whenever } x_1 < x_2.$$

Prove that f is integrable.