MAT 257, Handout 15: January 16-20, 2012.

About the second Term Test.

The test will cover material corresponding to handouts 8 through 15. Thus, it will focus on integration, but may also touch on topics such as

- higher derivatives and Taylor's Theorem,
- the oscillation of a function (from Chapter 1 of Spivak, used in the proof of the theorem that characterizes integrable functions on a bounded rectangle in $\mathbb{R}^n$.)

The main results about integration are:

- definition and basic properties (linearity etc) of the Riemann integral. Easy criteria for integrability (e.g. $\forall \varepsilon > 0 \exists P$ such that $U(f, P) < L(f, P) + \varepsilon$.)
- partitions of unity, and definition and basic properties of the (extended) integral
- measure zero, content zero, Jordan measurable sets
- characterization of (Riemann) integrable functions
- Fubini's Theorem
- Change of Variables Theorem

You may need material from Chapters 1 and 2 such as compactness, the Chain Rule, Lemma 2-10, and the Implicit and Inverse Function Theorems. Unless specified otherwise, you can refer to anything was proved in class, in the homework, or in the handouts.

The samples of tests from past years contain pretty good practice problems. I will also try to post most problems later.

Some topics from last week's lectures.

Sard's Theorem: Here we followed quite closely the proof from Spivak (Theorem 3-14), with more detail supplied at certain places. Note that Spivak himself supplies more detail in the Addenda, on the very last page of the book.

Equivalent characterization of a manifold this was essentially Theorem 5-2 in Spivak, illustrated by pictures, and making one or two points a little more explicit.

Integral of a scalar function over a manifold. Suppose M is a k-dimensional manifold in $\mathbb{R}^n$, and $g : M \to \mathbb{R}$ is a function. We define the integral $\int_M g \, dV$ as described below. (Here the notation "dV" reminds us that we are integrating with respect to the "volume element" on M.)

1. first suppose that $f : W \to M$ is a coordinate system (ie, a map from an open subset $W \subset \mathbb{R}^k$ onto a subset of M, satisfying the conditions in the alternative characterization of a manifold, see Spivak's Theorem 5-2) and that g has compact support in $f(W)$. Then we define

$$\int_M g \, dV := \int_{f(W)} g \, dV = \int_W g \circ f \; V(f')$$

where

$$V(f')(x) \ := \ \sqrt{f'(x)^T f'(x)}.$$

(Note that the Jacobian matrix $f'(x)$ is a $n \times k$ matrix for every x, so that $f'(x)^T f'(x)$ is a $k \times k$ matrix, whose determinant is thus well-defined.) This definition makes sense whenever $g \circ f$ is integrable on W, which is certainly the case if g is continuous.

2. For a continuous function $g : M \to \mathbb{R}$, we proceed as follows.

First, we a *partition of unity for* M is defined to be the restriction to M of a partition of unity for an open neighbourhood (in $\mathbb{R}^n$) of M. We say that a partition of unity Φ is *admissible* if for every $\varphi \in \Phi$, there exists some coordinate system $f : W \to M$ such that φ is supported in $f(W)$. Then if g is any continuous function such that

$$\sum_{\varphi \in \Phi} \int_M |g\varphi| \ dV$$

converges absolutely for some admissible partition of unity, we define

$$\int_M g = \sum_{\varphi \in \Phi} \int_M g\varphi \ dV.$$

We then have

Theorem 1. *If $\sum_{\varphi \in \Phi} \int_M |g\varphi| \ dV$ converges absolutely, then $\int_M g \ dV$ is well-defined, in the sense that it is independent of the choice of (admissible) partition of unity Φ, as well as of the specific coordinate systems $f : W \to M$ that parametrize open sets in the open cover associated to Φ.*

The proof was discussed in the lecture. The proof of the fact that the integral is independent of the choice of partition of unity is very similar to the corresponding point in our treatment of extended integrals and was thus omitted. The proof of the other claim ultimately reduces to the change of variables theorem.

We also proved that if $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, then

$$\int_{\psi(M)} g \circ \psi^{-1} \ dV = \int_M g \ dV$$

for every integrable function $g : M \to \mathbb{R}$. The hypothesis means that $\psi'(x)^T \psi'(x) \equiv I$, and in fact implies that ψ has the form $\psi(x) = p + Ox$ for some $p \in \mathbb{R}^n$ and some matrix O such that $O^T O = I$.

Volume of a manifold. If M is a k-dimensional manifold in $\mathbb{R}^n$ and the constant function 1 is integrable on M, then we define the (k-dimensional) volume of M to be $v(M) := \int_M 1 \ dV$. If $k = 1$ or 2, we generally say "length" or "area"‘, respectively instead of "volume"