MAT 257, Handout 2: week of September 19-23, 2011.

I. Assignment.

Note that this is *not* to be handed in.

Over the next week: read the rest of Chapter 1 of Spivak, and reread the first 10 pages as necessary. Finish the problems on pages 4-5, work the problems on page 10, and start the problems on pages 13-14. (At least one of the problems on pages 10, 13-4 will appear on the next homework assignment.) Again, make sure to solve *in detail* all of the starred exercises. (However, for an exercise such as problem 19, a fully detailed solution may be only a sentence or two.)

Problem 18 on page 10 asks about open sets $A \subset [0,1]$ that contain every rational number in $(0,1)$. Note that these sets can be quite peculiar. For example, fix any $\varepsilon > 0$ and an enumeration $(q_i)_{i=1}^\infty$ of $\mathbb{Q} \cap (0,1)$, and let $A := (0,1) \cap \left(\cup_{i=1}^\infty (q_i - 2^{-i}\varepsilon, q_i + 2^{-i}\varepsilon)\right)$. This is an open set containing all the rationals in $(0,1)$. Note however that the intervals that comprise A have lengths that sum to at most ε, so (in this sense at least) A can be arbitrarily small.

Problems 20 and 21 are valid in any metric space. By contrast, problem 22 is *not* valid in arbitrary metric spaces. (There are metric spaces in which no nonempty open set is compact, so that every compact set has empty interior.)

II. Summary.

If X is any set, recall that a *metric* on X is any function $d : X \times X \to \mathbb{R}$ that is symmetric, positive definite, and satisfies the triangle inequality. (See the first homework assignment for details.)

A set X endowed with a fixed metric d is called a *metric space.*

We say that a subset U of a metric space X is *open* if for every $x \in U$, there exists an open ball B such that $x \in B \subset U$. (Recall that an *open ball* is a set of the form $\{x \in X : d(x,a) < r\}$, for some $a \in X$ and $r > 0$.) A subset A of X is defined to be *closed* if $X - A$ is open.

Theorem 1. *When $X = \mathbb{R}^n$ with the Euclidean metric, this definition of open is equivalent to the definition given in the textbook.*

Proof. First, we introduce the notation $|x|_{max} := \max\{|x^1|, \dots, |x^n|\}$. (One can check that this is a norm, as the notation suggests.) For any $x \in \mathbb{R}^n$, we will write $C_\varepsilon(x) := \{y \in \mathbb{R}^n : |x - y|_{max} < \varepsilon\}$ to denote the open cube of side-length 2ε centred at x (which is an open ball of radius ε with respect to the "max" metric.) We state an easy lemma, whose proof is left as an exercise.

Lemma 1. *For any $z \in \mathbb{R}^n$,*

$$|z|_{max} \le |z| \le \sqrt{n}|z|_{max}.$$

Taking this for granted, first assume that U is open in the above sense (i.e., using the definition with open balls). To show that it is open according to Spivak's definition, it suffices to fix an arbitrary $x \in U$, and to show that $C_\varepsilon(x) \subset U$ for some $\varepsilon > 0$.

Since U is open, we know that $x \in \{y \in \mathbb{R}^n : |y - a| < r\} \subset U$ for some $r > 0$ and $a \in \mathbb{R}^n$. Also, for any $y \in C_\varepsilon(x)$, the Lemma implies that $|y - x| \le \sqrt{n}|y - x|_{max} < \sqrt{n}\varepsilon$. Thus by Cauchy-Schwarz,

$$|y - a| \le |y - x| + |x - a| \le \sqrt{n}\varepsilon + |x - a|, \qquad \text{so} \qquad |y - a| < r \text{ if } \varepsilon < n^{-1/2}(r - |x - a|).$$

It follows that $C_\varepsilon(x) \subset \{y \in \mathbb{R}^n : |y - a| < r\} \subset U$ if $\varepsilon < n^{-1/2}(r - |x - a|)$.

Conversely, assume that U is open according to Spivak's definition, and fix $x \in U$. We must now show that there exists an open (Euclidean) ball B such that $x \in B \subset U$. By hypothesis, there exists an open rectangle $R = (a^1, b^1) \times \cdots \times (a^n, b^n)$ such that $x \in R \subset U$. It follows that there exists some $\varepsilon > 0$ such that $x \in C_\varepsilon(x) \subset R$; indeed, this is true for $\varepsilon = \min\left(\{x^i - a^i\}_{i=1}^n \cup \{b^i - x^i\}\right)$. Then the Lemma implies that $B := \{y : |y - x| < \varepsilon\} \subset C_\varepsilon(x)$, since

$$y \in B \implies |y - x| < \varepsilon \overset{Lemma}{\implies} |y - x|_{max} < \varepsilon \implies y \in C_\varepsilon(x).$$

$\square$

We depart from the textbook by considering topology in the general setting of metric spaces rather than just in (Euclidean) $\mathbb{R}^n$. The above theorem guarantees that everything we will prove about topology of metric spaces applies in particular to the topology of $\mathbb{R}^n$ (as defined by Spivak).

We will also prove some facts that apply specifically to $\mathbb{R}^n$, most notably, a characterization of compact subsets of $\mathbb{R}^n$ (which is *not* valid for compact subsets of arbitrary metric spaces.) Apart from this, most of the proofs we will give of topological results are in fact very much the same in arbitrary metric spaces as in Euclidean space.

Examples of metric spaces include

- $\mathbb{R}^n$ with the Euclidean metric.
- $\mathbb{R}^n$ with the "max" metric $d_{max}(x,y) = |x-y|_{max}$.
- $\mathbb{R}^n$ with the "taxicab metric" $d_{sum}(x,y) = \sum |x^i - y^i|$.
- any set with the discrete metric, $d(x,y) := 0$ if $x = y$, and $d(x,y) := 1$ if $x \neq y$.

Our proof above shows that the first two spaces have the same topology (i.e., the same open sets), and a very similar proof (based for example on the inequality $|z| \le |z|_{sum} \le \sqrt{n}|z|$ for all $z \in \mathbb{R}^n$) shows that the third space also has the same topology.

A subset K of a metric space X is said to be *compact* if, for any collection $\mathcal{O} = \{O_\alpha\}_{\alpha\in A}$ of open subsets of X such that $K \subset \cup_{\alpha\in A} O_\alpha$, there exists a *finite* subset J of A such that $K \subset \cup_{\alpha\in J} O_\alpha$.

The notion of compactness is central to analysis. We spent a long time developing a list of examples of compact and non-compact spaces. Let me know if you can think of other interesting or illuminating examples!

III. Further exercises to think about.

(1) Show that if K is a compact subset of a metric space X, then for every $\varepsilon > 0$, K can be covered by a *finite* collection of open balls of radius at most ε.

(2) Assume that K is a compact subset of a metric space X, and that (x_m) is a sequence in K. Show that there exists a subsequence (x_{m_n}) and an element $x \in K$ such that $d(x_{m_n}, x) \to 0$ as $n \to \infty$.

Here are a number of easier special cases:

- $X = \mathbb{R}$, $K = [a,b]$ with $a < b$.
- $X = \mathbb{R}$, K is a closed (hence compact) subset of $[a,b]$ with $a < b$.
- $X = \mathbb{R}^n$, K an arbitrary compact set (and thus a closed subset of a closed rectangle.)

In the general case, you can proceed by finding some property of compact sets that would allow you to mimic, in at least some respects, the proof(s) you have found in various easier special cases. Of course, whether this last hint is helpful depends to some extent on which proofs you have found in the easier special cases.

(3) Let $x \mapsto |x|_*$ be a norm on $\mathbb{R}^n$.
   (a) Prove that $|\,|x+y|_* - |x|_*\,| \le |y|_*$ for all x and y in $\mathbb{R}^n$.
   (b) Prove that $|y|_* \to 0$ as $y \to 0$.
   (c) Conclude that the function $x \mapsto |x|_*$ is continuous as a function from $\mathbb{R}^n$ (with the standard Euclidean topology) to $\mathbb{R}$.

(4) Using the same notation as in the previous problem, prove that for any norm $|\cdot|_*$, there exist constants $c < C$ such that

$$m|x| \le |x|_* \le M|x| \quad \text{ for all } x \in \mathbb{R}^n.$$

Hint: Consider the behavior of $|x|_*$ for x such that $|x| = 1$, that is, points x on the Euclidean unit sphere.